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Some applications of coherent states in quantum electrodynamics

Dennis Leland Nordstrom
Iowa State University

SOME APPLICATIONS OF COHERENT STATES IN QUANTUM ELECTRODYNAMICS

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Dennis Leland Nordstrom

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In Charge of Major Work

Signature was redacted for privacy.

Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
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I. INTRODUCTION

Quantum electrodynamics has proven to be a highly successful theory of interacting fields. Even though the theory has been beset with divergence difficulties, predictions from the theory have been found to be in excellent agreement with experiment. These divergences have been satisfactorily resolved by various means. One such means is to remove the infinite zero-point quantities, e.g., infinite zero-point energy, by slightly modifying the formal theory by requiring that all operators representing physical observables be normal-ordered. Another type of divergence, the ultraviolet divergence, is handled by a renormalization procedure which consists of isolating the divergence from finite physical quantities and redefining physical constants in the theory, for example, the electron mass and charge. A third type of divergence, the infrared divergence, has also been shown to cause no serious difficulties; yet its removal lacks a completely satisfactory physical justification and is therefore the basis for the discussion presented here.

In the following treatment it will be assumed that all physical quantities have been properly renormalized so that the usual Feynman-Dyson perturbation techniques may be employed.

As an introduction, first consider the infrared divergence for real photons from a classical viewpoint. The semi-classical predictions of the infrared divergence phenomena may be understood by considering the deflection of a moving electron due to its interaction with a potential. The field of the electron is altered by the collision and the resulting change appears as emitted electromagnetic radiation. For wavelengths sufficiently

long in comparison to the dimensions of the scattering region and for negligible collision time the radiation will depend only on the initial and final momenta of the electron and the direction in which the radiation is observed. A well-known result (1, Sec. 69) is that in this limit the energy emitted per unit frequency is independent of frequency. In terms of photons the number of photons emitted per unit frequency range is inversely proportional to the frequency so that the photon spectrum is of the form $\frac{dk}{k}$ which diverges as $k \to 0$.

Semi-classical arguments also predict the angular distribution of the radiation. In the high energy limit the electron's field becomes contracted along the direction of motion. This leads to the direction of emission being confined to a narrow cone along either the initial or final direction of motion. This behavior is evident in the classical amplitude for emission of radiation which is proportional to

$$e\left(\frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k}\right) \tag{1}$$

where $\vec{p}$ and $\vec{p}'$ are the initial and final momenta, respectively, ϵ is the polarization vector for the emitted radiation whose wave number is k and e is the charge of the electron. The covariant notation used throughout this investigation is discussed in the appendix. The probability for emission of a photon of momentum $\vec{k}$ in the scattering process is then proportional to

$$e^2\left(\frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k}\right)^2 \frac{d^3k}{k} \tag{2}$$

This expression exhibits the infrared behavior and the strong tendency for emission of radiation parallel to either the initial or final momentum of the electron.

The same type of infrared behavior occurs in a complete quantum mechanical treatment of the emission of a low frequency real photon due to the scattering of an electron from a potential. A calculation of the matrix element for the emission process (2, p. 123) yields a result proportional to expression 1 times the basic matrix element for the scattering process with no emission. This result then displays the same characteristics as are found in the classical treatment of the emission of radiation, in particular, that the infrared divergence is associated with emission of soft photons from external charge lines only. A similar infrared behavior exists for the emission and reabsorption of soft virtual photons. In addition to the external line dependence the emission and absorption of low frequency photons does not cause a significant disturbance in the electron's motion. Therefore, the soft photons should be emitted and absorbed independently thus possessing a Poisson distribution. It is this similarity that is used to eliminate the infrared divergence problem when lowest order radiative corrections to a given process are made in the perturbative framework. Because of the zero mass of the photon and the limitations of the experimental apparatus there are always photons of sufficiently low energy emitted in a scattering process which escape detection. The calculated cross section for emission of an undetected photon then contains an infrared divergence which exactly cancels the infrared divergence due to the soft virtual photon corrections.

This then is the basic idea behind the resolution of the infrared divergence problem in the conventional treatment originally given by Bloch and Nordsieck (3). Since emitted photons of arbitrarily small energy may escape detection, the final state in any scattering experiment involving charged particles can never be completely known. By making several approximations, such as the neglect of pair effects and electron recoil effects, Bloch and Nordsieck showed that the cross section for emission of a finite number of photons in a scattering process is zero. This, of course, is due to the soft virtual photons. However, they were able to show that the probability to emit any number of soft photons sums to a non-zero result. They further showed that this result was very nearly equal to that obtained by ignoring all radiative corrections. This is due to the cancellation of the real and virtual infrared divergences as mentioned in the preceding paragraph.

After Bloch and Nordsieck solved the infrared divergence problem using several semi-classical approximations, a number of attempts were made to solve the problem employing a fully quantum field-theoretic treatment. Although several examples of the cancellation of real and virtual photons to lowest order may be found in the literature, two excellent works, among others, by Jauch and Rohrlich (4 and 5, Ch. 16) and Yennie, Frautschi, and Suura (6) present general treatments of the infrared divergence problem. These general treatments show how an additional soft real or virtual photon contribution may be isolated as a factor multiplying the basic matrix element for an arbitrary process. They then demonstrate how the cancellations are made in the cross sections.

The conventional treatment of the infrared divergence problem which consists of summing the cross sections over all possible final states prompted Chung (7) to conjecture that the divergence is quite probably due to the use of incorrect initial and final states because of the undetermined number of soft photons emitted. He then proposed letting these states be coherent states parametrized in such a way as to eliminate the infrared divergence in the matrix element for some arbitrary process. The coherent states are well known from quantum optics and are defined as the eigenstates of the photon annihilation operator. The coherent states were found to be quite appropriate since they allow for the presence of an unlimited number of photons. Storrow (8) also used the coherent states to demonstrate the possibility of formulating an S-matrix theory free of infrared divergences so that processes involving mixing of strong and electromagnetic interactions could be treated.

The purpose of this investigation is to give a physically satisfying interpretation and justification for the particular form of the coherent state used by Chung and Storrow. It is believed here that the infrared divergence is a spurious divergence since its source is not in the formulation of the Feynman-Dyson perturbation theory but rather is in its application. The reason for this is that in calculating a matrix element for some scattering process the virtual photon radiative corrections are integrated over all of momentum space. However, the conventional procedure is to consider only real emitted and absorbed photons with momenta above some small nonzero limit so that real photons with momenta ~ 0 are not included in the calculation. This minimum limit is a consequence of the limitations in resolution of the experimental apparatus and in

measurement techniques. Thus a correct application of the theory would be to include Feynman diagrams in which there are an arbitrary number of real emitted and absorbed photons with momenta ranging from zero to this minimum detectable limit. An alternative procedure would be to avoid consideration of low-frequency virtual photons, in addition to the real photons, by cancelling off that part of the electron self-field whose Fourier transform variable $\vec{k}$ lies below this minimum limit.

The most reasonable means of making this cancellation is by modifying the definition of the in- and out-electron states since an electron's momentum can never be known exactly either. The uncertainty in the electron momentum, due to the measurement process and the long-range Coulomb forces, indicates that a true description of the physical in- and out-electron is in terms of wave packets. Thus the cancellation of the low-frequency Fourier transformed self-field of the electron may be achieved by incorporating soft-photon contributions needed for the cancellation into the incoming and outgoing physical electron wave packet states and regulating the momentum spread of the electron by a weight function. The procedure then is to choose the wave packet states to be direct products of the coherent photon states, which can be used to describe low frequency photon fields, and the electron wave packet states described by electron creation operators acting on the vacuum. The cancellation is most easily made by considering the transverse self-field of the in- or out-electron as the classical field generated from the expectation value of the in- or out-current operator for the in- or out-electron state. Then the form for the coherent photon contribution to the wave packet state is determined by

demanding that for some fixed time the expectation value of the quantized electromagnetic in- or out-field for the wave packet state exactly cancel this classical self-field. Due to the lightlike velocity of the photons in the coherent state this cancellation cannot remain true at all times but will be so for frequencies arbitrarily close to zero which give rise to the divergence difficulties. By making the weight function more sharply peaked it is only these frequencies that are being described by the coherent state. The form for the physical electron wave packet state obtained in this manner is found to have exactly the same singularity structure in the coherent photon contributions as demanded by Chung and Storrow.

In Chapter II a review of the conventional treatment of the infrared divergence problem is given with the primary source being the article by Yennie, Frautschi and Suura. Chapter III presents a discussion of the definition and some of the properties of the coherent states as given by Glauber (9). The resolution of the infrared divergence problem by the coherent state treatment is demonstrated in Chapter IV. In Chapter V, which is the main original contribution reported in this thesis, the wave packet treatment is presented justifying the form for the coherent state contribution used by Chung and Storrow. Chapter VI presents conclusions, possible problems with the wave packet treatment, and suggestions for possible future research projects.

II. CONVENTIONAL TREATMENT OF INFRARED DIVERGENCE PROBLEM

The review of the conventional treatment of the infrared divergence problem in quantum electrodynamics presented here is taken from the article by Yennie, Frautschi, and Suura (6) (hereafter referred to as YFS). References to prior work performed on this problem are listed either in YFS or in the articles by Jauch and Rohrlich (4 and 5, Ch. 16). In addition to the general treatment given in the above-mentioned works, Nakanishi (10) has presented a treatment which includes infrared divergences due to massless photons and neutrinos and differs from YFS in that he restricts the treatment to the total cross section rather than the differential cross section.

Before presenting the conventional infrared divergence treatment, a listing of some of the more recently published works on topics dealing with soft photon effects will be given. Further references on these topics may be found in the articles listed.

An alternative approach to YFS has been suggested by Peres (11). He says the mass-shell conditions cannot be imposed on external charged particles because of the long-range electromagnetic forces. He then suggests relating $p^2 - m^2$ to the distance from the collision point which is then kept finite. The correspondence with the classical limit of bremsstrahlung amplitude for emission and absorption of an arbitrary number of soft photons has been shown by Brown and Goble (12). Belinfante (13) also relates to the classical limit by considering a semiclassical theory of infrared divergence which is essentially a simplified version of the Bloch-Nordsieck model. Lee and Nauenberg (14) demonstrate the cancellation of

infrared divergences in quantum electrodynamics by averaging over an appropriate ensemble of degenerate states in the power series expansions of the transition probabilities. The asymptotic condition in electrodynamics has been criticized by Hagen (15). As an alternative he introduced a nonlocal field operator which asymptotically creates physical electrons without the usual emission of soft quanta. He was then able to show how to calculate cross sections without the appearance of the infrared divergences.

A paper by Perrin and Lomon (16) shows how to incorporate the Bloch-Nordsieck approximation into the intermediate state sums in quantum electrodynamics. The infrared divergences can then be eliminated when approximations of cutting off intermediate state sums are made. Perrin (17) also uses this technique to study the infrared behavior of the wave function renormalization constant. Verbeure (18) has presented a discussion of the infrared divergent terms in the bound state picture. An article by Etim, Pancheri, and Touschek (19) shows the experimenter how to make infrared radiative corrections to a colliding beam experiment. In addition to the treatment of the infrared problem in the usual theory of quantum electrodynamics, Watanabe (20) discusses the problem as it appears in the perturbation series in the quantum statistical mechanics of an electron plasma. Weinberg (21) and Barker, Gupta, and Kaskas (22) also show that the infrared divergences arising in the quantum theory of gravitation due to the emission and absorption of soft massless gravitons can be removed by the same methods used in quantum electrodynamics.

A number of papers have been published dealing with soft photon

contributions in high energy processes, among them being those by Budini and Furlan (23), Furlan (24), Eriksson (25), Frautschi (26), and Soloviev (27).

Soft photon questions have also been examined through the use of the functional formulation of field theory as suggested by Schwinger (28) and discussed in the text by Bogoliubov and Shirkov (29). Using the functional integration method the infrared singularities of the Green's functions for various charged particles have been investigated by Hagen (30), Soloviev (31), Barbashov (32), Zhuravlev and Soloviev (33), and Jackiw and Soloviev (34), the latter also treating infrared gravitons. This technique has also been used by Barbashov and Volkov (35) to eliminate infrared singularities in scattering cross sections by employing the Bloch-Nordsieck idea. Tarski (36) has presented an operator solution for the Bloch-Nordsieck model in the formulation of Boboliubov and Shirkov and discussed the infrared question in terms of that solution. Soloviev and Yushin (37) have also presented a discussion of the infrared singularities of the matrix elements in scalar electrodynamics by employing the renormalization group methods, also used by Eriksson above.

The following presentation of the treatment of YFS employs the Dirac theory notation as used by Bjorken and Drell (2 and 38).

The conventional treatment of the infrared divergence problem will be demonstrated by considering an electron potential scattering process which is simple yet illustrates the main features of the treatment. The procedure will be to factor out the infrared contributions and identify them with well-defined Feynman diagrams. The real and virtual soft photon contributions will be considered separately with the virtual photon

corrections being looked at first.

Suppose an electron in a state of momentum p scatters from an external potential to a final state of momentum p' with a fixed number of photons in the initial and final states. By considering radiative corrections the matrix element for this process is

$$M(p,p') = \sum_{n=0}^{\infty} M_n (p,p') \tag{3}$$

where M_n is the contribution corresponding to all Feynman diagrams in which there are n virtual photons distinguishable from the potential interactions in the basic process M_0. The real photon variables are suppressed.

Let M_n be defined by

$$M_n = \frac{1}{n!} \int \cdots \int \prod_{i=1}^{n} \frac{d^4k_i}{k_i^2 - \lambda^2} \rho_n (k_1 \cdots k_n) \tag{4}$$

where the photon mass λ is allowed to approach zero later. The factor $\frac{1}{n!}$ is due to symmetrization of the n virtual photons. To extract the infrared contribution to ρ_n due to the nth virtual photon consider the Feynman diagrams for which there are n-1 virtual photons. The only diagrams giving an infrared contribution due to k_n are those in which both ends of the nth virtual photon terminate on external charge lines. All other diagrams in which at least one end of the nth virtual photon terminates on an internal line are finite as $k_n \to 0$ as long as all other virtual photon momenta k_i are nonzero. If $k_n \to 0$ and $k_i \to 0$ simultaneously, then overlapping divergences arise. However, these cancel when individually nongauge-invariant terms are combined into gauge-invariant expressions.

To see how the extraction goes let $\Gamma(p,p')$ represent the shaded area in Fig. 1 corresponding to all potential interactions, the n-1 virtual photons and emission and absorption of any real photons. Then with the momentum space Feynman propagator for an internal fermion line given as $i(\not{p} - m + i\eta)^{-1}$ the diagram in Fig. 2 for the radiative correction due to the addition of the virtual photon k is proportional to

$$\bar{u}(p')\gamma^{\mu}(\not{p}'-\not{k}-m)^{-1}\,\Gamma(p-k,p'-k)\,(\not{p}-\not{k}-m)^{-1}\,\gamma_{\mu}\,u(p)$$

$$= \frac{(2p-k)\cdot(2p'-k)}{(k^2-2p\cdot k)(k^2-2p'\cdot k)}\;\bar{u}(p')\Gamma(p,p')u(p) \qquad (5)$$

$$+ \frac{(2p-k)\cdot(2p'-k)}{(k^2-2p\cdot k)(k^2-2p'\cdot k)}\;\bar{u}(p')\,[\Gamma(p-k,p'-k) - \Gamma(p,p')]\,u(p) + K(k)$$

where the equality follows from the properties of the Dirac spinors. The spin description has been suppressed in the Dirac spinors so that $u(p,s) = u(p)$. $K(k)$ is not infrared divergent in k and the infrared divergence in the remaining photon momenta has not been made worse. By careful consideration of overlapping divergences the factor in brackets in the second term contributes a finite amount as $k \to 0$ and therefore belongs to $K(k)$.

The other diagrams contributing an infrared divergence due to k_n are those for which k_n gives electron self-energy parts on the external charge lines and are the usual wave function renormalization diagrams. They will be proportional to Equation 5 with p replaced by p' and vice versa for the two cases. Therefore, when all these infrared divergent terms are included,

ρ_n is of the form

$$\rho_n(k_1 \cdots k_n) = S(k_n)\ \rho_{n-1}\ (k_1 \cdots k_{n-1}) + \beta_n^{(1)}\ (k_1 \cdots k_{n-1}; k_n) \qquad (6)$$

where the infrared contribution due to k_n is contained entirely in $S(k_n)$ which has the form

$$S(k) = \frac{ie^2}{2(2\pi)^4} \left(\frac{2p'_\mu - k_\mu}{2p' \cdot k - k^2} - \frac{2p_\mu - k_\mu}{2p \cdot k - k^2} \right)^2 \qquad (7)$$

The term $\beta_n^{(1)}$ does not lead to an infrared divergence in k_n and its infrared divergences in the other k_i's has not been made worse by the separation.

The ultraviolet divergences, which arise from self-energy and vertex parts of Feynman diagrams are associated with the mass and charge renormalization. The mass renormalization is handled in the usual way by cancellation with an added mass counter term. The spurious charge renormalization is removed by grouping the diagrams so their infrared parts may be identified, then cancelled. Closed charge loops cause no difficulties since it can be shown that they yield contributions which are noninfrared divergent in k except for vacuum polarization loops which give the proper charge renormalization.

By repeated iterations of Equation 6 and by exploiting the symmetry in ρ_n, the expression for ρ_n can be written as a sum over all permutations of k_i and k_j:

$$\rho_n(k_1 \cdots k_n) = \sum_{\text{perm}} \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \prod_{i=1}^{r} S(k_i)\ \beta_{n-r}(k_{r+1} \cdots k_n) \qquad (8)$$

where the β_{n-r} are not infrared divergent. On inserting Equation 8 into Equation 4 the matrix element for the existence of n virtual photons becomes

$$M_n = \sum_{r=0}^{n} \frac{1}{r!\,(n-r)!} \left(\int \frac{d^4k\ S(k)}{k^2 - \lambda^2} \right)^r \int \cdots \int \prod_{i=1}^{n-r} \frac{d^4k_i}{k_i^2}\, \beta_{n-r}(k_1 \cdots k_{n-r}) \tag{9}$$

The fictitious photon mass λ has been dropped in the last factor since it is no longer needed to prevent an infrared divergence. By defining

$$\alpha B = \int \frac{d^4k\ S(k)}{k^2 - \lambda^2} \tag{10}$$

and

$$m_r(p,p') = \frac{1}{r!} \int \cdots \int \prod_{i=1}^{r} \frac{d^4k_i}{k_i^2}\, \beta_r(k_1 \cdots k_{n-r}) \tag{11}$$

so that

$$M_n = \sum_{r=0}^{n} \frac{1}{r!}\, (\alpha B)^r\, m_{n-r} \tag{12}$$

the resulting expression for the matrix element of Equation 3 takes the following simple form

$$M(p,p') = \exp(\alpha B) \sum_{n=0}^{\infty} m_n \tag{13}$$

By definition, m_n is an infrared divergence-free function, with $m_0 = \beta_0 = \rho_0 = M_0$, so that the entire infrared divergence due to the virtual photons has been isolated in the exponential. It may also be noted that

the integral representation for α_B as defined in Equation 10 converges as $k^2 \to \infty$ so that no ultraviolet cutoff is needed.

The infrared contribution to α_B arises from the poles in the momentum space photon propagator,

$$\frac{1}{k^2 - \lambda^2 + i\eta} = P \frac{1}{k^2 - \lambda^2} - i\pi\delta(k^2 - \lambda^2) \tag{14}$$

and gives the real part of α_B:

$$\mathrm{Re}(\alpha_B) = \frac{e^2}{4(2\pi)^3} \int \frac{d^3k}{(k^2 + \lambda^2)^{1/2}} \left(\frac{2p'_\mu - k_\mu}{2p' \cdot k - \lambda^2} - \frac{2p_\mu - k_\mu}{2p \cdot k - \lambda^2} \right)^2 \tag{15}$$

The extraction of the infrared contributions due to emission or absorption of soft real photons goes through in a similar manner. The infrared contributions are again due to external line emission and absorption only. Let $\Gamma(p,p')$ again represent the shaded area in Fig. 1. Then the diagrams for emission of a real photon of momentum k and polarization ϵ, as in Figs. 3 and 4, are proportional to

$$\bar{u}(p')\,\Gamma(p-k,p')(\not{p}-\not{k}-m)^{-1}\,\not{\epsilon}\,u(p) + \bar{u}(p')\,\not{\epsilon}\,(\not{p}'-\not{k}-m)^{-1}\,\Gamma(p,p'-k)u(p)$$

$$= -\left(\frac{p\cdot\epsilon}{p\cdot k} - \frac{p'\cdot\epsilon}{p'\cdot k} \right) \bar{u}(p')\,\Gamma(p,p')\,u(p) + K'(k) \tag{16}$$

where $K'(k)$ again is not infrared divergent in k. Thus the matrix element, represented by $\tilde{\rho}_n(k_1 \cdots k_n)$, for the emission or absorption of n undetectable real photons with momenta $k_1 \cdots k_n$ for some arbitrary order in the virtual photon corrections has the form

$$\tilde{\rho}_n\ (k_1 \cdots k_n) = \pm\, \tilde{S}(k_n)\ \tilde{\rho}_{n-1}(k_1 \cdots k_{n-1}) + \tilde{\beta}_n^{(1)}\ (k_1 \cdots k_{n-1}; k_n) \qquad (17)$$

where the (+) and (-) signs correspond to emission and absorption, respectively. The polarization index has been suppressed. For this case the infrared contribution due to the real photon k_n is contained entirely in the factor $\tilde{S}(k_n)$ which is given by

$$\tilde{S}(k) = \frac{e}{[2(2\pi)^3 k_o]^{1/2}} \left(\frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k} \right) \qquad (18)$$

where ϵ is the polarization vector for the photon of momentum k. The term $\tilde{\beta}_n^{(1)}$ again does not lead to an infrared divergence in k_n and its divergences in the other k's has not been made worse by the separation. Repeated iterations of Equation 17, along with the symmetry property of the $\tilde{\rho}_n$'s gives

$$\tilde{\rho}_n(k_1 \cdots k_n) = \sum_{\text{perm}} \sum_{r=0}^{n} (-1)^m \frac{1}{r!\,(n-r)!} \prod_{i=1}^{r} \tilde{S}(k_i)\tilde{\beta}_{n-r}(k_{r+1} \cdots k_n) \qquad (19)$$

where the $\tilde{\beta}_{n-r}$ are noninfrared and m corresponds to the number of absorbed real photons.

It should be noted that $\tilde{\rho}_n$ defined in Equation 17 differs from that in YFS in that their $\tilde{\rho}_n$ is a factor in the differential cross section; whereas $\tilde{\rho}_n$ above represents the matrix element. The two are related by essentially squaring Equation 18 to obtain $\tilde{S}$ of YFS. The reason for the departure from YFS in the discussion of soft real photon infrared contributions is that in using the coherent state techniques the infrared divergences will be shown to cancel in the matrix elements rather than in

the cross section as in the conventional treatment.

The more general scattering problem in which several electron lines interact with each other requires a generalization of some of the equations appearing above. Let an arbitrary process contain w charged incoming and outgoing particles, each with charge eZ_i and momentum p_i. The same diagrams as for the single charged particle line can be used where now the external lines represent any pair of free incoming or outgoing, boson or fermion, charged particles. The shaded area represents the remainder of the process including all other external lines and, as before, is independent of k. Then the generalization of αB from Equations 10 and 7 is given by

$$\alpha B = \sum_{i<j} \alpha B_{ij} = -\frac{ie^2}{2(2\pi)^4}\int \frac{d^4k}{k^2-\lambda^2} \sum_{i<j} Z_i\theta_i Z_j\theta_j \left(\frac{(2p_i\theta_i - k)_\mu}{k^2 - 2k\cdot p_i\theta_i} + \frac{(2p_j\theta_j + k)_\mu}{k^2 + 2k\cdot p_j\theta_j} \right)^2 \tag{20}$$

where Z_i has the sign of the i<u>th</u> charge and $\theta_i = +(-)$ if i is outgoing (incoming). Similarly, the generalization of $\tilde{S}(k)$ from Equation 18 is

$$\tilde{S}(k) = -\frac{e}{[2(2\pi)^3 k_o]^{1/2}} \sum_{i=1}^{w} \frac{Z_i\theta_i p_i\cdot\epsilon}{p_i \cdot k} \tag{21}$$

It is apparent from here that the infrared contributions for single electron potential scattering are a special case obtainable from Equations 20 and 21 by putting $Z_1 = Z_2 = -1$ and $\theta_1 = -\theta_2 = -1$.

It is interesting to note that as long as any of the terms in the sum in Equation 20 involves one incoming and one outgoing particle, those terms are always real. However, terms involving a pair of particles, both

incoming or outgoing, are imaginary. Thus there is a phase factor contribution to the matrix element which, by Equation 20, is infinite and gives the divergent Coulomb phase factor. The divergent phase arises from the singularities in the charged particle propagators, $(k^2 \pm 2k\cdot p_i \theta_i)^{-1}$; whereas the divergence in $\mathrm{Re}(\alpha B)$ comes from the singularities in the photon propagator, $(k^2 - \lambda^2)^{-1}$, as in Equation 14. Since it is the cross section that contains the physics of the process, phase factors are usually of little concern. In the coherent state treatment Storrow shows how these divergent phase factors may also be cancelled.

Although the summary of the relevant parts of the conventional treatment presented above is sufficient for use in the coherent state treatment, a brief outline of the remainder of the conventional treatment will be given for the sake of completeness. From Equation 13 the differential cross section for emission of n soft real photons with total energy Δ is

$$\frac{d\sigma_n}{d\Delta} = \exp(2\alpha B)\,\frac{1}{n!}\int\cdots\int\prod_{m=1}^{n}\frac{d^3k_m}{(k_m^2+\lambda^2)^{1/2}}\,\delta(\Delta-\Sigma k_i)\,\tilde{\rho}_{n\ YFS}(k_1\cdots k_n) \qquad (22)$$

Allowing for the presence of an unlimited number of soft real photons the complete differential cross section becomes

$$\frac{d\sigma}{d\Delta} = \lim_{\lambda\to 0}\sum_{n=0}^{\infty}\frac{d\sigma_n}{d\Delta} \qquad (23)$$

The extraction of real photon infrared contributions to $\tilde{\rho}_{n\ YFS}$ carried out in a manner analogous to the virtual photon case gives

$$\tilde{\rho}_{n\ YFS}(k_1 \cdots k_n) = \tilde{S}_{YFS}(k_n)\ \tilde{\rho}_{n-1\ YFS}(k_1 \cdots k_{n-1}) + \beta_n^{(1)}{}_{YFS}(k_1 \cdots k_{n-1}; k_n) \tag{24}$$

where

$$\tilde{S}_{YFS}(k) = -\frac{e^2}{2(2\pi)^3}\left(\frac{p'_\mu}{p' \cdot k} - \frac{p_\mu}{p \cdot k}\right)^2 \tag{25}$$

Putting it all together in a manner similar to Equations 6 through 13 finally gives

$$\frac{d\sigma}{d\Delta} = \lim_{\lambda \to 0} \exp\{2\alpha(B + \tilde{B})\} \frac{d\hat{\sigma}}{d\Delta} \tag{26}$$

where $\frac{d\hat{\sigma}}{d\Delta}$ is independent of the soft photon limit and

$$2\alpha\tilde{B} = \int \frac{d^3k}{(k^2+\lambda^2)^{1/2}}\ \tilde{S}_{YFS}(k) \tag{27}$$

The cancellation is made by comparing Equations 15 and 27.

III. COHERENT STATES

Although coherent states are of a type that have long been used in treatments of the harmonic oscillator problem, they have only recently gained widespread attention through their use in optical coherence theory and the infrared photon problem. One of the first uses of a type of coherent state was by Schrödinger (39) as he introduced wave packets having the same form as wave functions for the coherent states to illustrate the manner in which the oscillator approaches the classical limit. These states have also been used in the quantum theory of the harmonic oscillator to form a generating function for the usual occupation number states in the coordinate representation (40, p. 441).

The introduction of coherent states into quantized field theory would seem to be a natural one due to the mathematical equivalence of a quantized boson field and a system of independent harmonic oscillators each quantized according to the usual procedure of quantum mechanics. That this is indeed the case has been illustrated by Schwinger (41) in his series of papers on field quantization. In his discussion of the electromagnetic field he used the coherent states to generate the occupation number states in a manner analogous to the first quantized harmonic oscillator problem.

One of the earliest attempts to define the coherent states and examine some of their properties was presented by Klauder (42) when he introduced an overcomplete family of states, of which the coherent state is a member, in a discussion of the harmonic oscillator. Later, Klauder and McKenna (43) studied the function spaces spanned by overcomplete families of states in a series of discussions on the relation between quantum and classical dynamics from the standpoint of continuous represen-

tation theory. Recently, Rohrlich (44) has discussed the usefulness of a special case of the continuous representations, the coherent state representation, for the formulation of general quantum field theory and has indicated its value in demonstrating a relationship between corresponding quantum and classical field theories.

A fuller development of the properties of coherent states was made only after it was apparent that the usual occupation number states of the radiation field in quantum electrodynamics were very inconvenient states to use in describing the large and intrinsically uncertain number of photons in a given beam of light in optics. Since most optical fields had only a very low degree of coherence, this inconvenience was never very serious. However, with greatly improved techniques in the generation and detection of various types of correlations in optical fields, e.g., with the advent of the laser, it became apparent that a redefinition of the meaning of coherence was needed. Since the coherent states are of interest here and not the theory of quantum optics, only a brief survey of the correlation theory leading to the definition of coherent states will be given.

The earliest discussions of the quantum mechanical description of statistical light beams were given by Glauber (9 and 45) and Sudarshan (46) in which they defined coherent states and investigated some of their properties. The following presentation of the definition and properties of coherent states is based primarily on the article by Glauber. A thorough discussion of coherent states may also be found in the text by Klauder and Sudarshan (47).

In introducing the coherent states it is first assumed that the electric field operator $\vec{E}(x)$ can be separated into its positive and negative frequency parts, $\vec{E}^{(+)}(x)$ and $\vec{E}^{(-)}(x)$. The nth order correlation function of Glauber is then defined in terms of these individual fields as

$$G^{(n)}_{\mu_1\cdots\mu_{2n}}(x_1\cdots x_n,\ x_{n+1}\cdots x_{2n}) = \mathrm{Tr}\{\rho E^{(-)}_{\mu_1}(x_1)\ \cdots\ E^{(-)}_{\mu_n}(x_n)E^{(+)}_{\mu_{n+1}}(x_{n+1}) \cdots E^{(+)}_{\mu_{2n}}(x_{2n})\} \tag{28}$$

where Tr indicates the trace and ρ is the density operator for the field. A fully coherent state is then defined as one in which the nth order field correlation function factorizes into a product of 2n factors in the form

$$G^{(n)}_{\mu_1\cdots\mu_{2n}}(x_1\cdots x_{2n}) = \mathcal{E}^{*}_{\mu_1}(x_1)\cdots \mathcal{E}^{*}_{\mu_n}(x_n)\, \mathcal{E}_{\mu_{n+1}}(x_{n+1})\cdots \mathcal{E}_{\mu_{2n}}(x_{2n}) \tag{29}$$

This factorization is accomplished when the state of the field is an eigenstate of the positive- and negative- frequency parts of the electric field operator in the following sense

$$E^{(+)}_{\mu}(x)|\rangle = \mathcal{E}_{\mu}(x)|\rangle \tag{30}$$

$$\langle|E^{(-)}_{\mu}(x) = \mathcal{E}^{*}_{\mu}(x)\langle| \tag{31}$$

These eigenstates are the coherent states.

The usual procedure in a quantum field-theoretic treatment of the radiation field, as given by Bjorken and Drell (38, Ch. 14), is to take the vector potential $A^{\mu}(x)$ as the field operator and impose certain commutation relations between A^{μ} and the conjugate momenta. The electric

and magnetic field operators $\vec{E}(x)$ and $\vec{B}(x)$ are then derived from $A^{\mu} = (A^{o}, \vec{A})$ by the relations

$$\vec{E} = -\nabla A^{o} - \dot{\vec{A}} \quad , \quad \vec{B} = \nabla \times \vec{A} \tag{32}$$

For the free radiation field in the coulomb gauge, $\nabla \cdot \vec{A} = 0$, the time component of the four-vector potential may be set equal to zero. Then

$$\vec{E} = - \dot{\vec{A}} \tag{33}$$

The particle concept of the electromagnetic field emerges on expanding the potentials in plane waves and imposing certain commutation relations between the potentials and their conjugate momenta. Since the potentials in the radiation guage satisfy the free wave equation, $\Box \vec{A} = 0$, they possess the following momentum expansion

$$\vec{A}(x) = \sum_{\lambda=1}^{2} \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \hat{\epsilon}(k\,\lambda)[a(k\,\lambda)e^{-ik\cdot x} + a^{+}(k\,\lambda)e^{ik\cdot x}] \tag{34}$$

where $\hat{\epsilon}(k\,\lambda)$ is a unit polarization vector taken orthogonal to $\vec{k}$(so $\nabla\cdot\vec{A}=0$), $k_o = \omega = |\vec{k}|$, and $k^2 = k_\mu k^\mu = 0$. The operators $a(k\,\lambda)$ and $a^{+}(k\,\lambda)$ are interpreted as annihilation and creation operators of photons of momentum $\vec{k}$ and energy ω with the commutation relations

$$[a(k\,\lambda),\ a^{+}(k'\lambda')] = \delta_{\lambda\lambda'}\delta(\vec{k} - \vec{k}') \tag{35}$$

$$[a(k\,\lambda),\ a(k'\lambda')] = [a^{+}(k\,\lambda),\ a^{+}(k'\lambda')] = 0 \tag{36}$$

In discussing the coherent states it is convenient to assume the radiation field is confined in a spatial volume of finite size. Then expansion of the vector potential in terms of a complete set of orthonormal vector mode functions, $\vec{u}_k(\vec{x})$, rather than the plane waves, yields

$$\vec{A}(x) = \sum_k \frac{1}{\sqrt{2\omega}} [a_k \vec{u}_k(\vec{x}) e^{-i\omega t} + a_k^+ \vec{u}_k^*(\vec{x}) e^{i\omega t}] \tag{37}$$

In this case the amplitude operators a_k and a_k^+ satisfy the commutation relations

$$[a_k, a_\ell^+] = \delta_{k\ell} \tag{38}$$

$$[a_k, a_\ell] = [a_k^+, a_\ell^+] = 0 \tag{39}$$

and are the well-known annihilation and creation operators from the quantum theory of the harmonic oscillator.

Using Equations 33 and 37 the positive frequency part of the electric field operator becomes

$$\vec{E}^{(+)}(x) = i \sum_k \sqrt{\frac{\omega}{2}}\, a_k \vec{u}_k(\vec{x})\, e^{-i\omega t} \tag{40}$$

From Equation 30 the eigenvalue functions $\vec{\mathcal{E}}(x)$ must also satisfy Maxwell's equations so that they too can be expanded in terms of the normal modes with the operator amplitude a_k replaced by a c-number coefficient in the form

$$\vec{\mathcal{E}}(x) = i \sum_k \sqrt{\frac{\omega}{2}}\, \alpha_k \vec{u}_k(\vec{x})\, e^{-i\omega t} \tag{41}$$

Thus, using the orthogonality property of the $\vec{u}_k$'s and the dynamical independence of the different modes, the states for the individual modes, by Equation 30, must obey the relation

$$a_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle \tag{42}$$

The coherent states of the field are then seen to be direct products of the individual states,

$$|\{\alpha_k\}\rangle = \prod_k |\alpha_k\rangle \tag{43}$$

Rocca and Sirugue (48) have presented an alternative definition of correlation functions by defining them in terms of the field potential operator $A^{\mu}(x)$. In doing so they run into gauge invariance problems. However, they are able to resolve these by using a smeared-out form of the field operator leading to correlation functionals without gauge invariance difficulties.

Although the term "coherent state" as used here applies only to those states which are eigenstates of the annihilation operator, Titulaer and Glauber (49) have shown that these are not the only ones which possess full coherence. However, they have also shown that the eigenstates of the annihilation operator differ from other states with full coherence in several respects. Two important differences are that the eigenstates of the annihilation operator are the only ones in which the variance of any annihilation operator vanishes and the only ones which have $\vec{\mathcal{E}}(x)$, in the definition of the correlation function in Equation 28, as the expectation value of $\vec{E}^{(+)}(x)$.

In order to discuss the properties of the coherent states it will be sufficient to consider only a single mode oscillator, i.e., one degree of freedom. Then the coherent states may be defined as eigenstates of the annihilation operator satisfying

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{44}$$

where α is some arbitrary complex number. The adjoint state vector then satisfies

$$\langle\alpha|a^{+} = \langle\alpha|\alpha^{*} \tag{45}$$

An equivalent definition of coherent states is as an expansion in terms

of the occupation number states. This definition is obtained by writing

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle \tag{46}$$

and finding an explicit expression for the expansion coefficients $\langle n|\alpha\rangle$ in terms of α and n. By using the following familiar properties of the $|n\rangle$ states

$$a|n\rangle = n^{1/2}|n-1\rangle \tag{47}$$

$$a^+|n\rangle = (n+1)^{1/2}|n+1\rangle \tag{48}$$

$$a^+a|n\rangle = n|n\rangle \tag{49}$$

along with Equation 44 a recursion relation for the coefficients $\langle n|\alpha\rangle$ is derived giving

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{1/2}}\ \langle 0|\alpha\rangle \tag{50}$$

Choosing the normalization such that $\langle\alpha|\alpha\rangle = 1$ the equivalent definition for $|\alpha\rangle$ becomes

$$|\alpha\rangle = \exp[-\tfrac{1}{2}|\alpha|^2]\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle = \exp[-\tfrac{1}{2}|\alpha|^2]\sum_n \frac{(\alpha a^+)^n}{n!}|0\rangle$$

$$= \exp[-\tfrac{1}{2}|\alpha|^2]\ \exp[\alpha a^+]|0\rangle \tag{51}$$

The adjoint state vector possesses a similar expansion

$$\langle\alpha| = \exp[-\tfrac{1}{2}|\alpha|^2]\sum_n \langle n|\frac{(\alpha^*)^n}{(n!)^{1/2}} = \exp[-\tfrac{1}{2}|\alpha|^2]\sum_n \langle 0|\frac{(\alpha^* a)^n}{n!}$$

$$= \langle 0|\exp[-\tfrac{1}{2}|\alpha|^2]\ \exp\ [\alpha^* a] \tag{52}$$

With this definition of the coherent state it is easy to see that the average occupation number of the nth state is given by the Poisson

distribution,

$$|\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2} \tag{53}$$

This particular property of the coherent states is an indication of their appropriateness in a treatment of the infrared divergence problem since, as was pointed out in the introduction, the soft photon emissions and absorptions should be statistically independent due to the unperturbed motion of the electron current source.

Coherent states can also be defined as the operation on the vacuum of a unitary operator which acts as a displacement operator on the annihilation and creation operators a and a^+. By denoting this unitary displacement operator by D and letting it be a function of some complex parameter β it then is chosen to satisfy

$$D^+(\beta)\, a\, D(\beta) = a + \beta \tag{54}$$

$$D^+(\beta)\, a^+\, D(\beta) = a^+ + \beta^* \tag{55}$$

If the coherent state is an eigenstate of the annihilation operator as in Equation 44, then Equation 54 can be used to show that $D^+(\beta)|\alpha\rangle$ is an eigenstate of a with eigenvalue $\alpha - \beta$. Thus $D^+(\alpha)|\alpha\rangle$ must be the vacuum state so that the coherent state may be defined by use of the unitary displacement operator as

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{56}$$

The explicit form for the displacement operator is obtained by considering infinitesimal displacements and using the commutation relations for the annihilation and creation operators with the result

$$D(\alpha) = \exp[\alpha a^+ - \alpha^* a] \tag{57}$$

By using the Baker-Hausdorff formula

$$\exp[A + B] = \exp[A] \exp[B] \exp\{-\tfrac{1}{2}[A, B]\} \tag{58}$$

valid for A and B any two operators whose commutator is a c-number, the displacement operator may be written in normal ordered form as

$$D(\alpha) = \exp[-\tfrac{1}{2}|\alpha|^2] \exp[\alpha a^+] \exp[-\alpha^* a] \tag{59}$$

This form facilitates comparison with Equation 51 since operation on the vacuum with Equation 59 gives

$$\begin{aligned} |\alpha\rangle = D(\alpha)|0\rangle &= \exp[-\tfrac{1}{2}|\alpha|^2] \exp[\alpha a^+] \exp[-\alpha^* a]\,|0\rangle \\ &= \exp[-\tfrac{1}{2}|\alpha|^2] \exp[\alpha a^+]\,|0\rangle \end{aligned} \tag{60}$$

Some interesting aspects of the coherent states may be demonstrated by considering their coordinate space and momentum space representations. Glauber (9) has shown that the wave packets so formed are displaced forms of the ground-state wave function of the harmonic oscillator and are the forms used to illustrate the correspondence with the classical limit. Another interesting feature of the wave packet formulation is that these wave functions possess the minimum uncertainty in localization of position and momentum allowable according to the uncertainty principle. In a paper discussing the possible states of a system after a simultaneous measurement of conjugate variables, such as coordinate and momentum, She and Heffner (50) have shown that the state of the system which possesses the minimum uncertainty after a simultaneous measurement is a coherent state corresponding to that representing a minimum uncertainty wave packet. In addition to the minimum uncertainty in position and momentum Carruthers and Nieto (51) have shown that for large N, where

N is the excitation number of an oscillator, the coherent states possess a minimum uncertainty in the products of N and phase angle as well. However, Jackiw (52) later showed that this is only an approximate minimum uncertainty and in fact is not a unique property of the coherent states alone. Using variational techniques he showed that the coherent states do not minimize the various uncertainty relations for number and phase operators. Nieto (53) has also shown that for large uncertainty in angular momentum, i.e., for large ΔL_z, the coherent states are also minimum uncertainty states in angular momentum and angle. For these reasons the coherent states are also referred to as minimum uncertainty or quasi-classical states.

Possibly the main reason for the limited use of the coherent states prior to their use in modern coherence theory and the infrared problem is that they are not orthogonal. When the expansions of the coherent states in terms of the occupation number states, Equations 51 and 52, are used, it is found that the scalar product of two states $|\alpha\rangle$ and $|\beta\rangle$ is

$$\langle\alpha|\beta\rangle = \exp[-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2] \sum_{m,n} \frac{(\alpha^*)^m \beta^n}{(m!\, n!)^{1/2}} \langle m|n\rangle \qquad (61)$$

But the occupation number states form an orthonormal set, $\langle m|n\rangle = \delta_{mn}$, so

$$\langle\alpha|\beta\rangle = \exp[-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2 + \alpha^*\beta] = \exp[-\tfrac{1}{2}|\alpha-\beta|^2 + i\ \mathrm{Im}(\alpha^*\beta)] \qquad (62)$$

Hence $|\alpha\rangle$ and $|\beta\rangle$ are clearly not orthogonal. Even though the coherent states are not orthogonal they do form a complete set so may be used as a set of basis states. Their completeness may easily be demonstrated by use of the identity

$$\int (\alpha^*)^n \alpha^m e^{-|\alpha|^2} d^2\alpha = \pi n! \delta_{mn} \tag{63}$$

where the element of area is taken as $d^2\alpha = d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha)$ and the integration is carried out over the entire complex α plane. Using this identity with Equations 51 and 52 yields

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_n |n\rangle\langle n| = 1 \tag{64}$$

Thus the coherent states do indeed form an overcomplete set. Cahill (54) has further shown that the coherent states $|\alpha_n\rangle$ are complete if α_n is any convergent sequence of complex numbers.

Now there must exist an expansion for an arbitrary state of an oscillator in terms of the occupation number states of the form

$$|\rangle = \sum_n c_n |n\rangle = \sum_n c_n \frac{(a^+)^n}{(n!)^{1/2}} |0\rangle \tag{65}$$

where $\sum_n |c_n|^2 = 1$ for proper normalization of the arbitrary state. This normalization permits the definition of an entire function of a complex variable z in the form

$$f(z) = \sum_n c_n \frac{z^n}{(n!)^{1/2}} \tag{66}$$

so that an aribtrary state corresponding to the function f(z) may be written as

$$|f\rangle = f(a^+)|0\rangle \tag{67}$$

Writing the coherent states in terms of related analytic functions allows an alternative means of discussing the states of an oscillator. In expanding the coherent states in terms of the $|n\rangle$ states it is well known that the latter span a separable Hilbert space H_1, where the subscript 1

indicates one degree of freedom. In the text of Klauder and Sudarshan (47) it is then shown that the coherent states formed by Equations 51 and 52 also span a separable Hilbert space C_1 which is isomorphic to H_1. The alternate description then consists of studying the entire analytic functions f(z) defined above which are taken as elements of a Hilbert space related to that spanned by the coherent states. These spaces have been discussed by Bargmann (55) and Segal (56) and are special cases of the function spaces studied by Klauder and McKenna (43).

The extension to an infinite number of degrees of freedom is necessary in order to describe the states of the field as in Equation 43. The occupation number states $|n_1 n_2 \ldots\rangle$ needed to discuss the field are simply the direct products of the single-degree-of-freedom states $|n\rangle$ and span a separable Hilbert space H_∞ which remains separable only if the number of particles, $\sum_{k=1}^{\infty} n_k$, remains finite. The Hilbert space H_∞ is then the direct product of the Hilbert spaces H_1 for all of the modes of the oscillator. By a similar generalization of the coherent states to an infinite number of degrees of freedom the coherent states of the field are taken to be direct products of the coherent states for individual modes and span a separable Hilbert space C_∞, which remains separable provided $\sum_k |\alpha_k|^2 < \infty$. To see that the restriction $\sum_k |\alpha_k|^2 < \infty$ implies a finite number of particles note that the number operator for a single mode, $N_k = a^+_k a_k$, with Equations 44 and 45 gives $N_k = |\alpha_k|^2$. Then the number of particles in the field is $\sum_k |\alpha_k|^2$ which, if finite, implies a finite number of particles.

The completeness relation in Equation 64 now makes it possible to use the coherent states as a basis set in which to make the expansion of an

arbitrary state as in Equation 67. Operating from the left on Equation 67 with Equation 64 yields the expansion

$$|f\rangle = \frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha|f(a^+)|0\rangle = \frac{1}{\pi}\int d^2\alpha |\alpha\rangle f(\alpha^*)\ \exp[-\tfrac{1}{2}|\alpha|^2] \qquad (68)$$

where the last equality follows from Equations 45 and 66. This expansion can be shown to be unique by inverting Equation 68 to obtain an explicit form for $f(\alpha^*)$ corresponding to the state vector $|f\rangle$. The inversion is made by operating from the left on Equation 68 with a coherent state, Equation 52, and using Equations 62 and 63 to get

$$f(\alpha^*) = \exp[\tfrac{1}{2}|\alpha|^2]\,\langle\alpha|f\rangle \qquad (69)$$

An expansion similar to Equation 68 exists for the adjoint state vector and is

$$\langle f| = \frac{1}{\pi}\int d^2\alpha\ \langle\alpha|[f(\alpha^*)]^*\ \exp[-\tfrac{1}{2}|\alpha|^2] \qquad (70)$$

This same procedure may be used to expand an arbitrary quantum mechanical operator T in terms of coherent state vectors. The resulting expansion is

$$T = \frac{1}{\pi^2}\iint |\alpha\rangle\ \tau\ (\alpha^*,\beta)\langle\beta|\ \exp[-\tfrac{1}{2}|\alpha|^2 - \tfrac{1}{2}|\beta|^2]\, d^2\alpha d^2\beta \qquad (71)$$

where

$$\tau(\alpha^*,\beta) = \sum_{n,m} T_{nm}\ (n!\ m!)^{-1/2}\ (\alpha^*)^n\ \beta^m \qquad (72)$$

T_{nm} represents the matrix elements of T connecting states with n and m quanta which for most quantum mechanical operators is dominated by certain finite powers of n and m. Thus the double series converges and represents

an entire function of the finite complex variables α^* and β. The resulting inversion of Equation 71 is

$$\tau(\alpha^*,\beta) = \langle\alpha|T|\beta\rangle \exp\left[\tfrac{1}{2}|\alpha|^2 + \tfrac{1}{2}|\beta|^2\right] \qquad (73)$$

It is the operator expansion in Equation 71 that is most useful in optical coherence theory since the particular operator of most interest is the density operator necessary for the description of arbitrary mixtures of quantum states. However, for the infrared problem this expansion is of little importance since the main usefulness of the coherent states comes from their defining equations, 44, 45, 51, and 52, and the state expansion in Equation 67. The arbitrary state expansion shows that the usual occupation number states may be expanded in terms of the coherent states.

The treatment of the coherent states and their properties presented above is for free electromagnetic fields so that there exists a simple time dependence for the annihilation operator in the Heisenberg picture of the form $a(t) = ae^{-i\omega t}$. The time dependence of the annihilation operator, hence the coherent state, for a system of harmonic oscillators in the presence of an interaction has been discussed by Glauber (57), Mehta and Sudarshan (58), Mehta, Chand, Sudarshan and Vedam (59) and Mista (60). They have obtained equations of motion restricting the form the annihilation operator may take if an initially coherent state is to remain coherent at all times. They then derive the general form for the Hamiltonian subject to this requirement.

IV. COHERENT STATE TREATMENT OF INFRARED DIVERGENCE PROBLEM

The coherent state treatment of the infrared divergence problem was first presented by Chung (7) with later investigations by Greco and Rossi (61), Storrow (8), and Kibble (62). The treatments all show how the infrared divergences can be eliminated to all orders of perturbation theory in the matrix elements provided the initial and final states are taken to be the coherent soft-photon states of Glauber.

Chung's work was the first to suggest that the original infrared divergence in the matrix element could be attributed to the use of inappropriate initial and final states in an arbitrary scattering process. Since an indefinite number of soft photons are emitted in a scattering experiment it would seem that the usual occupation number states corresponding to a definite number of photons would not be reasonable physical states to use. Due to the classical nature of the soft-photon emission process and the need to describe an undetermined number of soft photons the coherent states, which have quasi-classical properties as pointed out in Chapter III, are used as the appropriate initial and final states and are parametrized in such a way that the calculation of the matrix element yields no infrared divergences. The matrix elements are calculated by using the usual Feynman-Dyson perturbation techniques which demand that the photon be given an artificial finite mass which can only be set to zero at the end of the calculations. Although the matrix elements so calculated are free of infrared divergences they still contain infinite phase factors for processes with several interacting electron lines.

The treatment by Greco and Rossi is essentially the same as that of

Chung except that they use coherent states to describe the final states but leave the initial states with a definite number of photons. Although this procedure still leads to a cancellation of infrared divergences, it would appear to be unsuitable for multiple scattering processes where intermediate states are used as initial states for one calculation and as final states for another. Their method for determining the form of the coherent state is to define new final states as the operation by a phase factor operator on the usual final state describing a finite number of particles. The form of the operator phase is chosen to be that of an interaction Hamiltonian for the quantized electromagnetic field interacting with a classical current source whose Fourier transform has the singularity structure needed for the infrared cancellations.

By extending the coherent soft-photon state description to include the initial state Storrow has shown that an S-matrix theory of electromagnetic interactions free of infrared divergences is possible. His procedure is to note that since perturbation theory calculations yield infrared divergences only for Feynman diagrams in which the soft photons are connected to external charge lines a good approximation would be to assume a classical current source as used by Greco and Rossi. Then the S-matrix connecting the in- and out-states takes a simple form as shown by Bjorken and Drell (38, Sec. 17.10). The particular form of the Fourier transformed c-number current source is then taken to be that of a charged particle receiving an instantaneous acceleration. This is reasonable because of the dependence of soft photon effects on external charge lines only so that the details of the acceleration process are unimportant. The resultant form

for the coherent state is then the same as that used by Chung except for the manner in which they restrict the possible coherent states by confining the undetectable soft photons to some "resolution region."

The treatment by Kibble is more mathematically rigorous than the others and takes a different approach to the problem. The treatments mentioned above assume the S-matrix elements to be the basic quantities which are calculated by summing the relevant Feynman diagrams using coherent states as appropriate initial and final states. Kibble's procedure is to use Schwinger's functional formulation of field theory to determine the possible asymptotic states of the system.

The wave packet treatment of the infrared problem presented in Chapter V uses the Feynman diagram summation procedure of Chung and Storrow. Therefore their work will be emphasized in the following presentation followed by a short summary of Kibble's treatment.

To demonstrate the cancellation of infrared divergences by using coherent state techniques Chung first considers the interaction of a single electron line with a potential to all orders and later extends the treatment to include the interactions of several electron lines. The coherent soft-photon states are taken to be generalizations of the single mode states in Equations 51 and 52 to an infinite number of degrees of freedom so as to describe the entire electromagnetic field. This generalization is indicated in Equation 43 and is

$$|\{\alpha_i\}\rangle = \prod_i |\alpha_i\rangle \tag{74}$$

with

$$|\alpha_i\rangle = \exp[-\tfrac{1}{2}|\alpha_i|^2]\ \exp[\alpha_i a_i^+]\ |0\rangle \tag{75}$$

The parametrization of the coherent states is then made by letting the index i represent an abbreviation for the variables λ and a where λ represents what will become a polarization index and a is used to denote a particular member of a complete and orthonormal set of functions $\{f_a(k)\}$ defined on some region Ω of momentum space including k = 0. Then have

$$\alpha_i \equiv \alpha_a^\lambda \quad , \qquad a_i^+ \equiv a_a^+(\lambda) \tag{76}$$

Next let

$$a_i^+ \equiv a_a^+(\lambda) = \int_\Omega d^3k\ f_a(k)a^+(k\lambda) \tag{77}$$

where $a^+(k\lambda)$ is the in-photon creation operator with commutation relations given in Equations 35 and 36. Then the coherent soft-photon in-state for the scattering of an electron from a state of momentum p to momentum p' is taken to be

$$|\{\alpha_a^\lambda\}\rangle = \prod_{a,\lambda} \exp[-\tfrac{1}{2}|\alpha_a^\lambda|^2]\ \exp[\alpha_a^\lambda \int_\Omega d^3k\ f_a(k)a^+(k\lambda)]\ |0\rangle$$

$$= \prod_a \exp[-\tfrac{1}{2}\sum_\lambda |\alpha_a^\lambda|^2]\ \exp[\sum_\lambda \alpha_a^\lambda \int_\Omega d^3k\ f_a(k)a^+(k\lambda)]\ |0\rangle \tag{78}$$

The coherent state expression for the final state can be written in a similar manner as

$$|\{\gamma_a^\lambda\}\rangle = \prod_a \exp[-\frac{1}{2}\sum_\lambda|\gamma_a^\lambda|^2]\ \exp[\sum_\lambda \gamma_a^\lambda \int_\Omega d^3k\ f_a(k)a^+(k\lambda)]|0\rangle \qquad (79)$$

The particular parametrization of the coherent states that is needed to secure the infrared cancellation is such as to require that α_a^λ and γ_a^λ be defined in the following way:

$$\alpha_a^\lambda = \beta_{ia}^\lambda + \epsilon_{ia}^\lambda \quad , \qquad \gamma_a^\lambda = \beta_{fa}^\lambda + \epsilon_{fa}^\lambda \qquad (80)$$

where the coefficients β_{ia}^λ and β_{fa}^λ are defined by

$$\beta_{ia}^\lambda = \int_\Omega d^3k\ f_a^*(k)\ \tilde{S}_i(k\lambda) \quad , \quad \beta_{fa}^\lambda = \int_\Omega d^3k\ f_a^*(k)\ \tilde{S}_f(k\lambda) \qquad (81)$$

with

$$\tilde{S}_i(k\lambda) = \frac{e}{[2(2\pi)^3k_o]^{1/2}}\ \frac{p\cdot\epsilon(k\lambda)}{p\cdot k} \quad , \quad \tilde{S}_f(k\lambda) = \frac{e}{[2(2\pi)^3k_o]^{1/2}}\ \frac{p'\cdot\epsilon(k\lambda)}{p'\cdot k} \qquad (82)$$

The functions $\tilde{S}_i$ and $\tilde{S}_f$ are familiar from the conventional treatment since by Equation 18

$$\tilde{S}(k\lambda) = \tilde{S}_f(k\lambda) - \tilde{S}_i(k\lambda) \qquad (83)$$

In the following the region of integration in k-space will always be over Ω.

To see how the cancellation goes consider the diagrams represented by Fig. 5 for a single electron line interacting with a potential in which

there are ℓ noninteracting soft real photons, m soft real photons absorbed by the electron line and m' soft real photons emitted by the electron line. All virtual photon lines are included in the shaded area. Then the matrix element for scattering from the initial state $|\{\alpha_a^\lambda\}, p\rangle$ to the final state $|\{\gamma_a^\lambda\}, p'\rangle$ is obtained by evaluating such a diagram and summing over all values of ℓ, m and m'. Since the calculation of the matrix element is rather messy, only the contributions to the matrix element, with the initial and final coherent states as in Equations 78 and 79, will be given and the final result simply written down.

From Equation 13 there is a contribution due to the virtual photon corrections of the form

$$\exp(\alpha B) \tag{84}$$

The contribution due to the overlap of the ℓ initial-state noninteracting photons with the ℓ final-state noninteracting photons is

$$\ell! \Big[\sum_{\substack{a,c \\ \lambda,\lambda'}} \alpha_a^\lambda \gamma_c^{\lambda'*} \iint d^3k\, d^3k'\, f_a(k) f_c^*(k') \delta_{\lambda\lambda'} \delta(\vec{k} - \vec{k}') \Big]^\ell = \ell! \Big[\sum_{a,\lambda} \alpha_a^\lambda \gamma_a^{\lambda*} \Big]^\ell \tag{85}$$

where the orthonormality of the $f_a(k)$'s is used. From Equation 19 and the form of the coherent state expressions in Equations 78 and 79 there is a contribution of the form

$$\Big[\prod_{r=1}^{m} \sum_{a,\lambda} \alpha_a^\lambda \int d^3k_r\, f_a(k_r) \Big] \Big[\prod_{r'=m+1}^{m+m'} \sum_{c,\lambda'} \gamma_c^{\lambda'*} \int d^3k_{r'} f_c^*(k_{r'}) \Big]$$

$$\times \tilde{\rho}_{m+m'}(k_1 \ldots k_{m+m'}) \tag{86}$$

where

$$\tilde{\rho}_{m+m'}(k_1 \ldots k_{m+m'}) = \sum_{\text{perm}} \sum_{t=o}^{m+m'} (-1)^m \frac{1}{t!(m+m'-t)!}$$

$$\times \prod_{i=1}^{t} \mathcal{S}(k_i \lambda_i)\, \tilde{\beta}_{m+m'-t}(k_{t+1} \ldots k_{m+m'}) \qquad (87)$$

The contributions from the normalizing factors in the coherent state expressions have not been included in 85 and 86 but can easily be seen to yield the factors

$$\frac{1}{(m+\ell)!} \quad \frac{1}{(m'+\ell)!} \exp\left[-\frac{1}{2} \sum_{a,\lambda} |\alpha_a^\lambda|^2\right] \exp\left[-\frac{1}{2} \sum_{a,\lambda} |\gamma_a^\lambda|^2\right] \qquad (88)$$

A final combinatorial factor accounting for the number of ways that $m+\ell$ initial-state photons and $m'+\ell$ final-state photons can be distributed among m initial-state, m' final-state interacting photons and ℓ noninteracting photons gives the final contribution of

$$\frac{(m+\ell)!}{m!\ \ell!} \quad \frac{(m'+\ell)!}{m'!\ \ell!} \qquad (89)$$

The final expression for the matrix element M obtained by putting expressions 84 through 89 together and summing over m, m' and ℓ is

$$M = \exp(\alpha B)\ \exp[-\frac{1}{2}\sum_{a,\lambda}|\alpha_a^\lambda|^2]\ \exp[-\frac{1}{2}\sum_{a,\lambda}|\gamma_a^\lambda|^2]\ \exp[\sum_{a,\lambda}\alpha_a^\lambda\,\gamma_a^{\lambda*}]$$

$$\times\ \exp[-\sum_{a,\lambda}\alpha_a^\lambda\int d^3k f_a(k)\tilde{S}(k\lambda)]\ \exp[\sum_{a,\lambda}\gamma_a^{\lambda*}\int d^3k f_a^*(k)\tilde{S}(k\lambda)]\sum_{m,m'=o}^{\infty} m_{m,m'} \tag{90}$$

where the last sum is term by term divergence-free. By using Equations 80, 81, and 83 this can be written in the form

$$M = \exp(\alpha B)\ \exp\{\sum_{a,\lambda}[\frac{1}{2}|\beta_{fa}^\lambda - \beta_{ia}^\lambda|^2 - \frac{1}{2}|\epsilon_{fa}^\lambda - \epsilon_{ia}^\lambda|^2]\}\ \exp(i\Phi)\sum_{m,m'=o}^{\infty} m_{m,m'} \tag{91}$$

where Φ is real and given by

$$\Phi = \sum_{a,\lambda} \mathrm{Im}(\beta_{ia}^{\lambda*}\,\epsilon_{ia}^\lambda + \beta_{fa}^\lambda\,\epsilon_{fa}^{\lambda*} - \beta_{ia}^\lambda\,\beta_{fa}^{\lambda*} + \epsilon_{ia}^\lambda\,\epsilon_{fa}^{\lambda*}) \tag{92}$$

Equations 81 and 83 allow the first term in the second exponential to be written as

$$\frac{1}{2}\sum_{a,\lambda}|\beta_{fa}^\lambda - \beta_{ia}^\lambda|^2 = \frac{1}{2}\sum_{a,\lambda}\int d^3k f_a(k)\tilde{S}(k\lambda)\int d^3k' f_a^*(k')\tilde{S}(k'\lambda)$$

$$= \frac{1}{2}\sum_{\lambda}\int d^3k\,|\tilde{S}(k\lambda)|^2 \tag{93}$$

where the reality of $\tilde{S}(k\lambda)$ and the completeness of the $f_a(k)$'s are used. Using the following relation for performing a polarization sum

$$\sum_{\lambda} g(k\lambda)\cdot\epsilon(k\lambda)\ h(k\lambda)\cdot\epsilon(k\lambda) = -\,g(k\lambda)\cdot h(k\lambda) \tag{94}$$

valid for $k\cdot g = k\cdot h = 0$, and Equation 18, Equation 93 becomes

$$\frac{1}{2}\sum_{a,\lambda}|\beta^{\lambda}_{fa} - \beta^{\lambda}_{ia}|^2 = \frac{e^2}{4(2\pi)^3 k_o}\sum_{\lambda}\int d^3k\left[\frac{p'\cdot\epsilon(k\lambda)}{p'\cdot k} - \frac{p\cdot\epsilon(k\lambda)}{p\cdot k}\right]^2$$

$$= -\frac{e^2}{4(2\pi)^3 k_o}\int d^3k\left[\frac{p'_\mu}{p'\cdot k} - \frac{p_\mu}{p\cdot k}\right]^2 = \alpha\tilde{B} \qquad (95)$$

where the last equality follows from Equations 25 and 27. Then the matrix element in Equation 91 becomes

$$M = \exp(\alpha B + \alpha\tilde{B})\exp[-\frac{1}{2}\sum_{a,\lambda}|\epsilon^{\lambda}_{fa} - \epsilon^{\lambda}_{ia}|^2]\exp(i\Phi)\sum_{m,m'=o}^{\infty} m_{m,m'} \qquad (96)$$

Comparison of Equations 95 and 15 shows that the infrared divergence which occurs when the photon mass is taken to zero cancels in the matrix element M rather than in the cross section as is the case in the conventional treatment. By restricting the possible states of the system by the condition

$$\sum_{a,\lambda}|\epsilon^{\lambda}_{fa} - \epsilon^{\lambda}_{ia}|^2 < \infty \qquad (97)$$

the matrix element M describing the scattering is a nonzero quantity free of infrared divergences.

Now it was pointed out in Chapter III that a coherent state of the field of the form

$$|\{\alpha_i\}\rangle = \prod_i \exp[-\frac{1}{2}|\alpha_i|^2]\exp[\alpha_i a_i^+]\,|0\rangle \qquad (98)$$

belongs to a separable Hilbert space only if

$$\sum_i |\alpha_i|^2 < \infty \tag{99}$$

This was also shown to mean that the number of quanta had to be finite. By rewriting Equations 80 as

$$\gamma_a^\lambda = \alpha_a^\lambda + \beta_{fa}^\lambda - \beta_{ia}^\lambda + \epsilon_{fa}^\lambda - \epsilon_{ia}^\lambda \tag{100}$$

it can be shown that the inequality 97 demands that the coherent states for the initial and final states of the scattering process cannot both belong to the separable Hilbert space C_∞. That is, to have a nonzero matrix element there must be an infinite number of soft photons in either the initial state or the final state. This, of course, is the essence of the Bloch-Nordsieck solution. To see how this goes suppose the initial state satisfies the condition given in 99,

$$\sum_{a,\lambda} |\alpha_a^\lambda|^2 < \infty \tag{101}$$

From Equation 95

$$\sum_{a,\lambda} |\beta_{fa}^\lambda - \beta_{ia}^\lambda|^2 \to \infty \tag{102}$$

as the photon mass approaches zero. Thus if the matrix element M in Equation 96 is to be nonzero, i.e., if condition 97 holds, then from Equation 100

$$\sum_{a,\lambda} |\gamma_a^\lambda|^2 \to \infty \tag{103}$$

Therefore there must be an infinite number of particles in the final state so that $|\{\gamma_a^\lambda\}>$ cannot belong to the separable Hilbert space C_∞ but rather must belong to a space which is unitarily inequivalent to C_∞. This means that the coherent states belong to representations of the canonical

commutation relations which are unitarily inequivalent to the usual Fock representation. For a mathematically rigorous discussion of the coherent states and their associated spaces see the papers by Kibble.

In order to make this theory physically acceptable it is first necessary to give an interpretation and justification for the form of the coherent states used. This problem will be taken care of by the wave packet treatment in Chapter V. Secondly, it is necessary to place a restriction on the region Ω of momentum space on which the functions $\{f_a(k)\}$ are defined. From the coherent state expressions in Equations 78 and 79 it is apparent that the photons in the coherent state have momentum k belonging to Ω. Since these photons are soft, hence unobservable, it becomes necessary to restrict Ω to include only those momenta which cannot be detected by the experimental apparatus. The region Ω is then referred to as the "resolution region". That is, photons which escape detection belong to Ω and are called "soft" while detectable photons do not belong to Ω and are labelled "hard". Chung also shows that the infrared divergences still cancel for the case where the initial state and final state resolution regions differ. The only change for this latter case is that an additional convergence condition is needed to ensure finite nonzero matrix elements.

Storrow produces an infrared cancellation with very nearly the same procedure as above including the definition of the resolution region. His approach differs in that he chooses to parametrize the coherent states in a slightly different manner. Chung's parametrization of the incoming coherent state from Equations 78, 80 and 81 is such that

$$\sum_a \alpha_a^\lambda f_a(k) = \tilde{S}_i(k\lambda) + \sum_a \epsilon_{ia}^\lambda f_a(k) \qquad (104)$$

whereas Storrow uses essentially the following parametrization

$$\sum_a \alpha_a^\lambda f_a(k) = \tilde{S}_i(k\lambda)\, \Psi(p,k) \tag{105}$$

where $\Psi(p,k)$ has the value 1 at $|\vec{k}| = 0$ and is only nonzero in some neighborhood of $|\vec{k}| = 0$ which is taken to be the resolution region Ω. The outgoing coherent state is parametrized in a similar manner. It is actually Storrow's parametrization that will result from the wave packet treatment to be demonstrated in Chapter V.

The generalization of the coherent state treatment to the case of several interacting electron lines has been given by Chung and follows the procedure for treating the single electron potential scattering case except for a huge increase in the number of indices. An interesting aspect of the generalized case is the appearance of infinite phase factors due to the virtual photon corrections as mentioned in Chapter II and given by $\exp[i\,\mathrm{Im}(\alpha_B)]$ where α_B is as in Equation 20. Although Chung neglected to treat this, Storrow shows how these infinite phase factors may be cancelled by multiplying the n-particle state, written as the direct product of the single electron states each with its own accompanying coherent state, by an appropriate phase factor. The effect of this phase factor multiplication is essentially to give distorted Coulomb waves instead of the usual plane waves. That the phase factor cancellation is not automatic in Chung's treatment is evident from the discussion following Equation 22. Since divergent phases arise from singularities in charged particle propagators and therefore correspond to off-mass-shell photons, the coherent state contributions cannot be expected to produce a cancellation since they describe on-mass-shell photons only.

Finally, an abbreviated summary of Kibble's treatment using variational derivative techniques will be given for completeness. The procedure is to make a rigorous definition of generalized coherent states of the radiation field which span a nonseparable Hilbert space. In order to make this definition the states must be given an additional label corresponding to an infinite phase factor. This is essentially a mathematical justification for the phase-factor treatment of Storrow. Kibble prefaces his treatment by considering the interaction of the quantized electromagnetic field with a prescribed classical current distribution. The S-matrix element between two generalized coherent states is determined by performing a variational derivative with respect to the external current and then solving the resulting equation. The expression for the matrix element so obtained is shown to be similar to Chung's results. In the extension to the fully quantized theory he shows how it is possible to determine the structure of the space of asymptotic soft-photon states by examining the singularity structure of the Green's functions which are assumed to contain the complete information about the theory. This is done by identifying a "core" Feynman diagram, that is, one with no soft-photon lines, and inserting the soft-photon lines in all possible ways. The insertion process is accomplished by the functional differentiation technique for interaction with an external soft-photon field. Once the soft-photon contribution to the Green's function is obtained its mass-shell singularities are studied to determine the nature of the asymptotic soft-photon coherent states in the absence of any massive particles or hard photons. These asymptotic states are found to have the properties of the generalized

coherent states spanning the nonseparable Hilbert space mentioned earlier. He then goes on to consider states containing one, and later several, massive particles in addition to the soft-photon coherent state. By using the LSZ reduction technique he finally calculates scattering matrix elements between the asymptotic soft-photon coherent states and shows that all the matrix elements are finite, including phases.

V. WAVE PACKET TREATMENT OF INFRARED DIVERGENCE PROBLEM

A correct calculation of a matrix element in quantum electrodynamics for an arbitrary scattering process using perturbation theory should lead to sensible results. The application of the theory as formulated consists of summing all Feynman diagrams corresponding to a given set of incoming and outgoing particles. Since the photon has zero mass it is not possible to specify the actual number of incoming and outgoing photons for a calculation to all orders in the coupling constant. This fact is the basis of the arguments used in the usual treatments of the infrared divergence problem as demonstrated in the preceding chapters. That there can never be initial and final scattering states with a finite number of particles with sharp momentum can be shown in two ways.

The first way is from the mathematics in the calculation process and was first shown by Bloch and Nordsieck(3). By assuming a finite number of photons in the initial and final states the matrix element calculated to all orders is zero. However, when allowing for the presence of any number of photons, it has been shown in Chapters II and IV that the matrix element can be nonzero and finite indicating that there must be an unlimited number of photons in the initial and/or final state.

A second way of demonstrating the existence of an arbitrary number of photons is much less rigorous but perhaps more physically appealing. It consists of a crude application of the phenomenon of diffraction dissociation (63 and 64). Assume an incoming electron is a bare electron accompanied by its self-field represented by virtual photons. Then let the electron undergo diffraction scattering so a small amount of momentum

may be transferred (with essentially no energy loss) to a virtual photon with arbitrarily small momentum $\vec{k}$ corresponding to the Fourier transform variable of the electron's self-field. Since the amount of momentum transferred is completely arbitrary it may be enough to kick this photon onto its mass-shell in which case it escapes as a real photon of arbitrarily small energy. Furthermore, since the self-field may be built up of an arbitrary number of these, there may be an unlimited number escaping as real photons thus giving an arbitrary number of photons in the final state . Going further with the diffraction dissociation approximation, it may be that some of these virtual photons are not kicked onto their mass-shells but remain as virtual photons. In this case they may be considered as being accounted for in the sum over diagrams allowing for any number of virtual photon corrections to the matrix element. Therefore, when the matrix element is calculated to all orders in the coupling constant, these soft virtual photons are taken into account by integrating over all possible values of momentum for the internal lines. For a consistent application of the perturbation theory then,all possible values of the external photon line momenta should also be considered. This means it must be assumed that there are an arbitrary number of particles in the initial and final states so the in- and out-states must be members of a Hilbert space which is unitarily inequivalent to the familiar occupation number space.

Now there are essentially two approaches that can be taken toward making a correct application of the perturbation theory. The first approach is that of the conventional treatment and coherent state treatment of the infrared problem. That is, an attempt may be made to

allow for an arbitrary number of photons in the scattering process. This is done in the conventional treatment by calculating the cross section for emission of a fixed number of soft photons by using the usual occupation number states then making an infinite sum over all such cross sections and integrating over the region of phase space available to these undetected photons. The total momentum and energy of the undetected photons is fixed and depends on the particular experimental situation. The coherent state treatment is an improvement in the sense that it is not the occupation number states that are being used but rather states which can describe the arbitrary number of soft photons possible thus belonging to a different Hilbert space. However, there is an arbitrariness in the choice of parametrization of the coherent states since the coefficients β_{ia}^{λ} and β_{fa}^{λ} defined in Equations 81 and restricting the form of the coherent states by Equations 78, 79 and 80 may equally well describe an arbitrary number of soft photons with different choices of $\tilde{S}_i$ and $\tilde{S}_f$ in Equations 82. The only requirement for description of an unlimited number of soft photons is that $\tilde{S}_i$ and $\tilde{S}_f$ be singular in the required manner as $k \to 0$. The particular choice of parametrization indicated in Equations 82 has the additional feature of leading to elimination of the infrared divergences in the matrix element.

An alternative procedure for making a correct perturbation theory calculation is to avoid all consideration of the troublesome soft real and virtual photons within experimental reason. This would be valid since the soft photons are undetectable and should not disturb the excellent agreement between theory and experiment. This may be done by noting that the diffraction dissociation approximations treat the soft

virtual photon corrections to the matrix element in the perturbation expansion as manifestations of the low-frequency Fourier transformed self-field of the electron. Therefore, if only real photons with momenta above some minimum detectable limit are to be considered, it is necessary that this part of the electron's self-field be cancelled so that no real or virtual photon with momentum $\vec{k}$ below this minimum limit is considered. A means will now be demonstrated for making this cancellation by using wave packets to describe in- and out-electrons.

The conventional procedure in scattering theory is to describe the initial and final states as free-particle states which are isolated from the interaction. This isolation may be accomplished in a mathematically convenient way by using the adiabatic hypothesis to localize the interaction in time (2, Chapters 6 and 9). An alternative procedure is to build wave packets localized in space which do not overlap the interaction region before or after the scattering thus representing free incoming and outgoing particles. From the uncertainty principle, of course, the momentum of a particle can never be known exactly so that a wave packet description is necessary. Then the correct asymptotic condition in field theory describes the asymptotic behavior of the matrix element of a field operator for any two normalizable states. That is, it expresses the initial and final conditions on these localized wave packets representing the incoming and outgoing particles. The states formed by the in-field creation operators are always to be understood as plane wave limits of normalizable wave packet states. It is convenient to use the wave packet states to cancel the low frequency part of the electron's self-field by modifying the definition of the in-

and out-electron states. The modification needed will be to incorporate soft-photon contributions by redefining the in- and out-states as direct products of coherent states and occupation number states. Since there are an arbitrary number of soft virtual photons in the perturbation expansion, the modified wave packet state must have a soft-photon contribution corresponding to an arbitrary number of soft photons. Thus the new wave packet states must belong to a Hilbert space unitarily inequivalent to the occupation number space.

To construct the wave packet states first recall that the fermion in-field Ψ_{in} satisfies the free Dirac equation

$$(i \not\partial - m)\, \Psi_{in}(x) = 0 \tag{106}$$

where the physical mass of the free incoming electron is represented by m. The particle interpretation is demonstrated by carrying out the quantization in momentum space by writing the solution of Equation 106 as

$$\Psi_{in}(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{\frac{m}{E}}} \left[b_{in}(ps)u(ps)e^{-ip\cdot x} + d^{+}_{in}(ps)v(ps)e^{ip\cdot x}\right] \tag{107}$$

with $E = p_o = (|\vec{p}|^2 + m^2)^{1/2}$. (The subscript "in" will be dropped in all of the following as all operators will be assumed to be in-operators.) The quantized amplitudes b(ps) and d(ps) are annihilation operators for electrons and positrons of four-momentum p and spin s, respectively, and satisfy the following anticommutation relations

$$\{b(ps),\ b^{+}(p's')\} = \delta_{ss'}\delta(\vec{p} - \vec{p}') \tag{108}$$

$$\{d(ps),\ d^{+}(p's')\} = \delta_{ss'}\delta(\vec{p} - \vec{p}') \tag{109}$$

$$\{b(ps),\ b(p's')\} = \{d(ps),\ d(p's')\} = 0 \tag{110}$$

$$\{b(ps),\ d(p's')\} = \{b(ps),\ d^{+}(p's')\} = 0 \qquad (111)$$

Therefore the state of a free incoming electron with momentum p and spin s is given by operating on the vacuum with the creation operator $b^{+}(ps)$,

$$b^{+}(ps)|0\rangle \qquad (112)$$

The soft-photon cancellation factor must now be included in the wave packet state obtained by giving some momentum spread to the in-electron state in expression 112. Because of the classical nature of the soft-photon emission process it seems reasonable that the soft-photon contribution to this wave packet state should be described by coherent states which have quasi-classical properties as discussed in Chapter III. Thus the wave packet states describing the incoming physical electron may be chosen to be direct products of the coherent photon states describing low-frequency photon fields and the electron wave packet states of expression 112. A particular state denoted by $|p\ s\ in\rangle$ is then defined by

$$|p\ s\ in\rangle = \int d^{3}p'\,\rho(p,\ p')N(p')\ \exp\{\sum_{\lambda}\int d^{3}k\ f(p'k\lambda)a^{+}(k\lambda)\}b^{+}(p's)|0\rangle \qquad (113)$$

where $\rho(p,\ p')$ is a weight function which will be left unspecified with the only restriction being that it fall off quite rapidly for p' differing from p. The fall-off behavior of the weight function may be chosen to correspond to a given experimental situation. In the ideal situation of a sharp momentum electron the weight function approaches a delta function. Since there is no electron spin dependence associated with soft-photon effects, as evidenced in Equations 6, 7, 17 and 18, the coherent photon contribution represented by the exponential function in Equation 113 may

be assumed to be indpendent of the spin s. The factor $N(p')$ has been inserted to allow the normalization to be independent of the coherent photon contributions. With the wave packet state defined as in Equation 113 the inner product of two such states is

$$\langle p's' \text{ in}|p\ s \text{ in}\rangle = \iint d^3p_1 d^3p_2\ \rho^*(p',\ p_1)\rho(p,p_2)N^*(p_1)N(p_2)$$

$$\times\langle 0|b(p_1s')\exp\{\sum_\lambda \int d^3k f^*(p_1k\lambda)a(k\lambda)\}\exp\{\sum_\lambda\int d^3k f(p_2k\lambda)a^+(k\lambda)\}b^+(p_2s)|0\rangle \tag{114}$$

By using the Baker-Hausdorff formula (Equation 58), Equations 35 and 108, $a(k\ \lambda)|0\rangle = 0$ and $b(ps)|0\rangle = 0$ it can be shown that Equation 114 reduces to

$$\langle p's' \text{ in}|p\ s \text{ in}\rangle = \delta_{ss'}\int d^3p_1\rho^*(p',p_1)\rho(p,p_1)|N(p_1)|^2\ \exp\{\sum_\lambda\int d^3k|f(p_1k\lambda)|^2\} \tag{115}$$

Therefore choose

$$N(p) = \exp\{-\frac{1}{2}\sum_\lambda\int d^3k|f(p\ k\ \lambda)|^2\} \tag{116}$$

so that

$$\langle p's' \text{ in}|p\ s \text{ in}\rangle = \delta_{ss'}\int d^3p_1\ \rho^*(p',p_1)\rho(p,\ p_1) \tag{117}$$

Then let the wave packet states be normalized to one,

$$\langle p\ s \text{ in}|p\ s \text{ in}\rangle = \int d^3p_1|\rho(p,\ p_1)|^2 = 1 \tag{118}$$

The formation of the wave packet state in Equation 113 gives states with properties similar to those used by Chung (7) and Storrow (8). That is, the asymptotic states are taken to be direct products of coherent states and occupation number states. However, rather than arbitrarily choosing

the parametrizing function $f(pk\lambda)$, it will now be determined from the cancellation requirement.

The cancellation is most easily made by considering the self-field of the incoming electron as the classical field generated from the expectation value of the in-current operator for the in-state electron. Then for some fixed time the requirement will be made that the expectation value of the quantized electromagnetic in-field for the wave packet state exactly cancel this classical self-field. In the Coulomb gauge the vector potential is transverse and from Maxwell's equations satisfies the wave equation

$$\Box \vec{A}(x) = \vec{j}^{tr}(x) \tag{119}$$

If the in-electron is described by the quantized in-current operator $\vec{j}(x)$ then

$$\vec{j}(x) = e\bar{\Psi}(x)\,\vec{\gamma}\,\Psi(x) \tag{120}$$

where e is the physical electron charge since Equation 120 defines the in-current. Then the classical self-field of the in-electron generated from the expectation value of the in-current operator in Equation 120 for the wave packet state is given by

$$\int d^4y\, D_{ret}(x-y)\langle p\ s\ in|\vec{j}^{tr}(y)|p\ s\ in\rangle \tag{121}$$

where $D_{ret}(x-y)$ is the retarded Green's function of the zero-mass Klein-Gordon equation. The explicit form for D_{ret} has the following Fourier representation as given by Roman (65, Sec. 1.2)

$$D_{ret}(x-y) = -\frac{1}{(2\pi)^4}\int d^4k\ \frac{e^{ik\cdot(x-y)}}{k^2-ik_o\eta} \tag{122}$$

where η is a small positive real constant, and the integration is

strictly along the real axis. After doing the integration the limit $\eta \to +0$ can be performed. It is easy to see that expression 121 gives the classical self-field of the electron since it is a particular solution of the classical wave equation

$$\Box\langle p\ s\ in|\vec{A}(x)|p\ s\ in\rangle = \langle p\ s\ in|\vec{j}^{tr}(x)|p\ s\ in\rangle \qquad (123)$$

This follows since from Equation 122

$$\Box D_{ret}(x-y) = \delta^4(x-y) \qquad (124)$$

An additional term corresponding to the solution of the homogeneous differential equation in Equation 123, i.e., for $\vec{j}^{tr}(x) = 0$, has not been included in Equation 121 since it represents the free electromagnetic field which is not the source of the divergence difficulties and may therefore be left alone.

It is now possible to make the cancellation by demanding that the expectation value of the quantized electromagnetic in-field for the wave packet state cancel the electron's classical self-field. That is, the proper parametrization of the coherent states is made by determining $f(p\,k\lambda)$ from the equation

$$\langle p\ s\ in|\vec{A}(x)|p\ s\ in\rangle = -\int d^4y D_{ret}(x-y)\langle p\ s\ in|\vec{j}^{tr}(y)|p\ s\ in\rangle \qquad (125)$$

where the minus sign secures the cancellation. Equation 125 does not say that the entire self-field will be cancelled but only the low-frequency Fourier transformed part which depends on the momentum spread allowed by the weight function $\rho(p, p')$.

With the wave packet state defined in Equation 113 and $\vec{A}(x)$ given

by Equation 34 the left hand side of Equation 125 (denoted by LHS) can be written as

$$\text{LHS} = \int\int d^3p'\, d^3p''\ \rho^*(p,p')\rho(p,p'')N(p')N(p')\langle 0|b(p's)\exp\{\sum_\lambda \int d^3k f^*(p'k\lambda)a(k\lambda)\}$$

$$\times\{\sum_\lambda \int \frac{d^3k}{\sqrt{2(2\pi)^3 k_o}}\ \hat{\epsilon}(k\lambda)[a(k\lambda)e^{-ik\cdot x}+a^+(k\lambda)e^{ik\cdot x}]\}\exp\{\sum_\lambda \int d^3k f(p''k\lambda)a^+(k\lambda)\}b^+(p''s)|0\rangle \tag{126}$$

The photon operators $a(k\lambda)$ and $a^+(k\lambda)$ commute with the fermion annihilation and creation operators so that Equation 126 reduces to

$$\text{LHS} = \int d^3p' |\rho(p,p')|^2 |N(p')|^2 \langle 0|\exp\{\sum_\lambda \int d^3k\ f^*(p'k\lambda)a(k\lambda)\}$$

$$\times\{\sum_\lambda \int \frac{d^3k}{\sqrt{2(2\pi)^3 k_o}}\ \hat{\epsilon}(k\lambda)[a(k\lambda)e^{-ik\cdot x}+a^+(k\lambda)e^{ik\cdot x}]\}\exp\{\sum_\lambda \int d^3k f(p'k\lambda)a^+(k\lambda)\}|0\rangle \tag{127}$$

By using the identity

$$e^B A\, e^{-B} = A + [B,\ A] \tag{128}$$

which is valid for $[A,B]$ = c-number, along with Equations 35, 36, 58 and 116, it is possible to rewrite Equation 127 as

$$\text{LHS} = \int d^3p' |\rho(p,p')|^2 \sum_\lambda \int \frac{d^3k}{\sqrt{2(2\pi)^3 k_o}}\hat{\epsilon}(k\lambda)[f(p'k\lambda)e^{-ik\cdot x}+f^*(p'k\lambda)e^{ik\cdot x}] \tag{129}$$

In equating this with the right hand side of Equation 125 it is easiest to consider the entire in-current operator $\vec{j}$ and later project out the transverse part. Thus, denoting the right hand side of Equation 125, with $\vec{j}^{tr}$ replaced by $\vec{j}$, as RHS, Equation 129 must be equal to the transverse part of

$$RHS = -\int d^4y D_{ret}(x-y)\langle p\ s\ in|\vec{j}(y)|p\ s\ in\rangle \tag{130}$$

When Equations 122, 113, 120 and 107 are used, this may be written as

$$RHS = \frac{e}{(2\pi)^4}\int d^4y \int d^4k\ \frac{e^{ik\cdot(x-y)}}{k^2 - ik_o\eta} \iint d^3p'\, d^3p''\, \rho^*(p,p')\rho(p,p'')N^*(p')N(p'')$$

$$\times \langle 0|b(p's)\exp\{\sum_\lambda \int d^3k f^*(p'k\lambda)a(k\lambda)\}$$

$$\times \{\sum_{\pm s_1} \int \frac{d^3p_1}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_1}}\,[b^+(p_1s_1)\bar{u}(p_1s_1)e^{ip_1\cdot y} + d(p_1s_1)\bar{v}(p_1s_1)e^{-ip_1\cdot y}]\}$$

$$\times \vec{\gamma}\{\sum_{\pm s_2} \int \frac{d^3p_2}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_2}}\,[b(p_2s_2)u(p_2s_2)e^{-ip_2\cdot y} + d^+(p_2s_2)v(p_2s_2)e^{ip_2\cdot y}]\}$$

$$\times \exp\{\sum_\lambda \int d^3k f(p''k\lambda)a^+(k\lambda)\}b^+(p''s)|0\rangle \tag{131}$$

The positron parts drop out from the normal-ordering requirement. Then with the commutation relations for the annihilation and creation operators and the Baker-Hausdorff formula it can be shown that Equation 131 reduces to

$$RHS = \frac{e}{(2\pi)^4}\int d^4y \int d^4k \frac{e^{ik\cdot(x-y)}}{k^2 - ik_o\eta} \iint d^3p'\, d^3p''\ \rho^*(p,p')\rho(p,p'')N^*(p')N(p'')$$

$$\times \exp\{\sum_\lambda \int d^3k f^*(p'k\lambda)f(p''k\lambda)\}\frac{1}{(2\pi)^3}\frac{m}{\sqrt{E'E''}}\ \bar{u}(p's)\ \vec{\gamma}\ u(p''s)e^{i(p'-p'')\cdot y} \tag{132}$$

The y integration may be done immediately yielding

$$RHS = \iint d^3p'\, d^3p''\ \rho^*(p,p')\rho(p,p'')N^*(p')N(p'')\exp\{\sum_\lambda \int d^3k f^*(p'k\lambda)f(p''k\lambda)\}$$

$$x \frac{e}{(2\pi)^3} \frac{m}{\sqrt{E'E''}} \bar{u}(p's) \vec{\gamma} u(p''s) \int d^4k \frac{e^{ik\cdot x}}{k^2 - ik_o\eta} \delta^4(k-p'+p'') \qquad (133)$$

Now do the k_o integration and let $\omega = |\vec{k}|$,

$$\int d^4k \frac{e^{ik\cdot x}}{k^2 - ik_o\eta} \delta^4(k-p'+p'') = \int d^3k \frac{e^{-i\vec{k}\cdot\vec{x}_e} i(p_o'-p_o'')x_o}{(p_o'-p_o'')^2 - \omega^2 - i(p_o'-p_o'')\eta} \delta(\vec{k}-\vec{p}'+\vec{p}'') \qquad (134)$$

The fraction in the integrand may be written as a sum of fractions by defining a new small positive constant ζ so that the right hand side of Equation 134 becomes

$$\int \frac{d^3k}{2\omega} \left[\frac{e^{i\vec{k}\cdot\vec{x}_e} i(p_o'-p_o'')x_o}{p_o''-p_o' - \omega + i\zeta} \delta(\vec{k}+\vec{p}'-\vec{p}'') + \frac{e^{-i\vec{k}\cdot\vec{x}_e} i(p_o'-p_o'')x_o}{p_o'-p_o'' - \omega - i\zeta} \delta(\vec{k} - \vec{p}' + \vec{p}'') \right] \qquad (135)$$

Therefore with Equation 116 and expression 135 it is possible to write

$$RHS = - \int\int d^3p' d^3p'' \; \rho^*(p,p')\rho(p,p'') \exp\left\{-\frac{1}{2} \sum_\lambda \int d^3k \left| f(p'k\lambda) - f(p''k\lambda) \right|^2 \right\}$$

$$x \exp\left\{ i \; Im \sum_\lambda \int d^3k \; f^*(p'k\lambda) f(p''k\lambda) \right\} \frac{e}{(2\pi)^3} \frac{m}{\sqrt{E'E''}} \bar{u}(p's) \vec{\gamma} u(p''s)$$

$$x \int \frac{d^3k}{2\omega} \left[\frac{e^{i\vec{k}\cdot\vec{x}_e} i(p_o'-p_o'')x_o}{\omega + p_o' - p_o'' - i\zeta} \delta(\vec{k}+\vec{p}'-\vec{p}'') + \frac{e^{-i\vec{k}\cdot\vec{x}_e} i(p_o'-p_o'')x_o}{\omega - p_o' + p_o'' + i\zeta} \delta(\vec{k}-\vec{p}'+\vec{p}'') \right] \qquad (136)$$

The comparison of Equation 136 with LHS of Equation 129 can be made by making a change of integration variables by defining

$$\vec{p}_1 = \frac{1}{2}(\vec{p}' + \vec{p}'') \;, \quad \vec{p}_2 = \vec{p}' - \vec{p}'' \qquad (137)$$

Then have $\rho(p,p') = \rho(p, p_1 + \frac{1}{2}p_2)$ where it is to be understood that this means $\vec{p}' = \vec{p}_1 + \frac{1}{2}\vec{p}_2$ and $E' = (|\vec{p}'|^2 + m^2)^{1/2} = (E_1^2 + \vec{p}_1 \cdot \vec{p}_2 + \frac{1}{4}|\vec{p}_2|^2)^{1/2}$. Making the substitutions of Equations 137 and doing the $\vec{p}_2$ integration, Equation 136 becomes

$$\text{RHS} = -\int d^3p_1 \int \frac{d^3k}{2\omega}\, \rho^*(p, p_1+\tfrac{1}{2}k)\rho(p, p_1-\tfrac{1}{2}k)\exp\{-\tfrac{1}{2}\sum_{\lambda'}\int d^3k' |f(p_1+\tfrac{1}{2}k, k', \lambda') - f(p_1-\tfrac{1}{2}k, k', \lambda')|^2\}$$

$$\times \exp\{i \operatorname{Im} \sum_{\lambda'} \int d^3k' f^*(p_1+\tfrac{1}{2}k, k', \lambda') f(p_1-\tfrac{1}{2}k, k', \lambda')\}$$

$$\times \frac{em}{(2\pi)^3}(E_1^2 + \vec{p}_1\cdot\vec{k} + \tfrac{1}{4}\omega^2)^{-\frac{1}{4}}(E_1^2 - \vec{p}_1\cdot\vec{k} + \tfrac{1}{4}\omega^2)^{-\frac{1}{4}}$$

$$\times \{\bar{u}(p_1-\tfrac{1}{2}k, s)\vec{\gamma}u(p_1+\tfrac{1}{2}k, s)\frac{e^{i\vec{k}\cdot\vec{x}_e}\, i[(E_1^2-\vec{p}_1\cdot\vec{k}+\frac{1}{4}\omega^2)^{\frac{1}{2}} - (E_1^2+\vec{p}_1\cdot\vec{k}+\frac{1}{4}\omega^2)^{\frac{1}{2}}]x_o}{\omega + (E_1^2-\vec{p}_1\cdot\vec{k}+\frac{1}{4}\omega^2)^{\frac{1}{2}} - (E_1^2+\vec{p}_1\cdot\vec{k}+\frac{1}{4}\omega^2)^{\frac{1}{2}} - i\zeta} + \text{c.c.}\} \tag{138}$$

where c.c. indicates that the second term in the braces is the complex conjugate of the first term. If the incoming electron has a fairly well-defined momentum, then the weight function $\rho(p,p')$ will be zero unless $p \sim p'$. From Equation 138 this means that

$$p_1 + \tfrac{1}{2}k \sim p \quad , \quad p_1 - \tfrac{1}{2}k \sim p \tag{139}$$

Therefore the sharp peaking requirement restricts k to values near zero, that is $k \sim 0$. In this case the following approximations may be made to determine $f(p\,k\,\lambda)$:

$$\rho(p,\ p_1 \pm \tfrac{1}{2}k) \rightarrow \rho(p,\ p_1) \tag{140}$$

$$\exp\{-\frac{1}{2}\sum_{\lambda'}\int d^3k' |f(p_1 + \frac{1}{2}k, k', \lambda') - f(p_1 - \frac{1}{2}k, k', \lambda')|^2\} \to 1 \tag{141}$$

$$\exp\{i\ \mathrm{Im}\sum_{\lambda'}\int d^3k' f^*(p_1 + \tfrac{1}{2}k, k', \lambda')\ f(p_1 - \tfrac{1}{2}k, k', \lambda')\} \to 1 \tag{142}$$

$$(E_1^2 \pm \vec{p}_1 \cdot \vec{k} + \tfrac{1}{4}\omega^2)^{-1/4} \to E_1^{-1/2} \tag{143}$$

$$\bar{u}(p_1 \pm \tfrac{1}{2}k, s)\ \vec{\gamma}\ u(p_1 \mp \tfrac{1}{2}k, s) \to \frac{\vec{p}_1}{m} \tag{144}$$

$$(E_1^2 \pm \vec{p}_1 \cdot \vec{k} + \tfrac{1}{4}\omega^2)^{1/2} \to E_1 \pm \frac{\vec{p}_1 \cdot \vec{k}}{2E_1} \tag{145}$$

Expression 144 follows from the properties of the Dirac spinors. The approximations made in 143 and 145 differ because 145 appears in Equation 138 as a difference of the two expressions in 145 so that $\frac{\vec{p}_1 \cdot \vec{k}}{2E_1}$ is the dominant factor while E_1 is dominant in 143 where there is no subtraction. Thus, when expressions 140 through 145 are substituted into Equation 138, RHS can be shown to approximately satisfy the equality

$$\mathrm{RHS} = -\int d^3p_1 |\rho(p, p_1)|^2 \int \frac{d^3k\ e}{2(2\pi)^3\omega}\ \frac{\vec{p}_1}{E_1}\ [\frac{e^{i(\vec{k}\cdot\vec{x} - \frac{\vec{p}_1\cdot\vec{k}}{E_1}x_o)}}{\omega - \frac{\vec{p}_1\cdot\vec{k}}{E_1} - i\zeta} + \mathrm{c.c.}] \tag{146}$$

Now Equation 129 must be equal to the transverse part of RHS. So to a first approximation the function $f(p\,k\lambda)$ must satisfy

$$\text{LHS} = -\int d^3p' |\rho(p,p')|^2 \int \frac{d^3k\, e}{2(2\pi)^3 \omega} \sum_\lambda \hat{\epsilon}(k\lambda)\vec{p}'\cdot\hat{\epsilon}(k\lambda)\left[\frac{e^{i(\vec{k}\cdot\vec{x} - \frac{\vec{p}'\cdot\vec{k}}{E'}x_o)}}{E'\omega - \vec{p}'\cdot\vec{k} - i\zeta E'} + \text{c.c.}\right] \tag{147}$$

Since the cancellation of the low-frequency Fourier transformed part of the electron's self-field is being made at some arbitrarily fixed time, it is convenient to choose this time to be $x_o = 0$. Then from Equations 129 and 147 it is a simple matter to solve for $f(p\,k\lambda)$ at the same time letting $\zeta \to +0$. The result is

$$f(p\,k\lambda) = -\frac{e}{\sqrt{2(2\pi)^3\omega}} \frac{\vec{p}\cdot\hat{\epsilon}(k\,\lambda)}{E\omega - \vec{p}\cdot\vec{k}} = \frac{e}{\sqrt{2(2\pi)^3 k_o}} \frac{p\cdot\epsilon(k\,\lambda)}{p\cdot k} \tag{148}$$

where the polarization four-vector is spacelike and in the present Lorentz frame has the components $\epsilon^\mu(k\,\lambda) = (0,\ \hat{\epsilon}(k\,\lambda))$. It is possible to obtain higher order corrections to $f(p\,k\lambda)$ with an iteration procedure by equating Equations 129 and 138. However, this will not be done here since Equation 148 is sufficient for cancellation of the infrared divergences. Since photons are massless, hence travelling with the velocity of light, this cancellation cannot be expected to be true for all time. But the limit k=0 causes the infrared divergence problem so this portion of the self-field should be cancelled for all time.

For a time different from zero, $x_o \neq 0$, the expression for $f(p\,k\lambda)$ obtained from equating Equations 129 and 147 differs from 148 by a phase factor $\exp\left[i\left(k_o - \frac{\vec{p}'\cdot\vec{k}}{E'}\right)x_o\right]$. However, in going to the ideal limit of sharp momentum by letting the weight function become a delta function, the only photons remaining in the coherent state description are those

with $k_\mu = 0$. For this case the phase factor reduces to a constant independent of time and equal to one. Then $f(p\,k\lambda)$ is given by Equation 148 for all time and the cancellation of the infrared divergences is made for all time.

Now the ideal situation mentioned above, whereby the electron's momentum is known exactly and all photon momenta are detectable, is never possible in practice, nor in principle according to the uncertainty principle. Rather, in a typical experiment there will always be some lower bound on detectable photon momenta. This same order of magnitude uncertainty exists in the determination of the momentum of the electron. Therefore an entirely reasonable procedure from the practical standpoint is to let the weight function have some nonzero momentum spread which may be chosen to correspond with some given experimental arrangement. This momentum spread will also regulate the magnitude of the region of phase space, by the relations in 139, which is available to the undetected real photons. This region may then be restricted to values of k_μ arbitrarily close to zero by tightening up the momentum spread. It might also be noted that for times different from zero the phase factor $\exp[i(k_o - \frac{\vec{p}\cdot\vec{k}}{E'})\, x_o]$ will go to one for $k_\mu \to 0$ so there will always be a cancellation of the infrared divergences.

An alternative way of defining $f(p\,k\lambda)$ is

$$f(p\,k\lambda) = g(p\,k\lambda)\Psi(p,\ k) \tag{149}$$

where

$$g(p\,k\lambda) = \frac{e}{\sqrt{2(2\pi)^3 k_o}} \quad \frac{p \cdot \epsilon}{p \cdot k} \tag{150}$$

and Ψ has the value 1 at k=0 and is only nonzero in some neighborhood of k=0 which may be taken to be the resolution region of Chung and Storrow. An explicit expression for Ψ could be obtained by an iteration solution from Equations 129 and 138. This resolution region may then be taken to be less than or equal to the momentum spread of the wave packet state.

The resulting expression for $f(p,k,\lambda)$ in Equations 149 and 150 is such as to produce a cancellation of the low-frequency Fourier transformed self-field of the electron by using asymptotic states as in Equation 113. These states are also found to have coherent photon contributions parametrized by functions with exactly the same singularity structure as those of Chung and Storrow. Therefore, by Chapter IV, the infrared divergences have been cancelled in the matrix elements.

VI. SUMMARY

This dissertation has presented an alternative solution of the infrared divergence problem in quantum electrodynamics. The conventional treatment is to use the Bloch-Nordsieck result which states that the cross section for emission of a finite number of photons in a given scattering process is zero. This null result is due to the infrared divergences in the virtual photon corrections when the cross section is calculated to all orders in the fine structure constant. Thus to obtain a nonzero result it is necessary to sum cross sections corresponding to Feynman diagrams allowing for any number of soft, undetectable real photons in the initial and final states which are taken to be the usual occupation number states. This unlimited number of soft photons may be described in a different way by the use of coherent states. In this case an incoming electron is assumed to be accompanied by a cloud of soft photons which is described by coherent states parametrized in such a way that the infrared divergences are eliminated in the matrix element.

The diffraction dissociation approximation has been used in Chapter V to show that the source of the infrared divergences is the electron's self-field whose Fourier transformed part represents off-mass-shell photons. In a diffraction scattering these soft virtual photons can either by kicked onto their mass-shells, then emerging as real photons, or remain as virtual photons. In a perturbation calculation then the latter photons are accounted for in the virtual photon corrections but the soft real photons, being undetectable, are not considered. Therefore the source of the infrared divergences is the low frequency self-field of the electron and arises

because of an inconsistent treatment of soft real and virtual photons. This approximation then indicates that an alternative solution of the infrared divergence problem is to cancel that portion of the low frequency self-field which causes infrared divergence problems but does not otherwise cause significant disagreements between theory and experiment.

A convenient way of making this cancellation has been demonstrated in Chapter V by incorporating soft-photon contributions into the usual wave packet states describing incoming particles. These modified wave packet states were taken to be direct products of coherent states, which describe the soft-photon contributions needed to cancel a portion of the electron's self-field, and the usual occupation number states for electrons. The new wave packet states are then capable of describing an arbitrary number of particles and therefore must belong to a Hilbert space unitarily inequivalent to the occupation number space. The parametrization of the coherent states was made by treating the self-field of the electron as a classical field generated from the expectation value of the in-current operator. The requirement was then made that this classical field should be cancelled by the expectation value of the quantized electromagnetic in-field. The resulting parametrization was found to be exactly that used by Chung and Storrow. Therefore, for a given scattering process the matrix element calculated by using these modified wave packet states as the correct asymptotic states will be free of infrared divergences.

The amount of momentum spread of the electron wave packet state, which is controlled by the peaking behavior of a weight function, may be chosen to correspond to a particular experimental arrangement. The width of this

wave packet may then be used to regulate the magnitude of the resolution region for undetectable photons. For instance, if the weight function goes to a delta function, then it is assumed that all photons of nonzero energy are capable of being detected. In practice, however, the wave packet will have a nonzero width so that the resolution region will contain other than just the $k_\mu = 0$ point of phase space.

This procedure has then provided a justification for the choice of asymptotic states used by Chung and Storrow by tying the source of the infrared divergence problem to the need to eliminate the low-frequency portion of the electron's self-field from consideration. The interpretation of the "cloud" of soft photons then is that it represents a low-frequency electromagnetic field which cancels part of the electron's self-field.

A possible problem arising from the wave packet treatment concerns what happens when the momentum spread of the electron is not sharply peaked. This would allow a larger range of values for the coherent photon momenta in which case the time factor in Equation 147 becomes significant. There will be no problem with infrared divergences, however, since the time factor still goes to one as k goes to zero. Rather, in this case there may be some difficulty in relating to experiment, if the resolution region is large enough, since there would be an uncertainty in what time is to be used. It's possible that this problem may be resolved by an iteration solution for $f(pk\lambda)$ obtained by equating Equations 129 and 138 thereby putting any time dependence into Ψ of Equation 149.

This question of a weight function not sharply peaked furnishes

one possible future research problem of interest since in actual experimental situations the momenta of the interacting particles is not always well-defined, in particular that of the outgoing particles. So by considering higher order approximations to $f(pk\lambda)$ by use of an iteration procedure it should be possible to determine the effects of a broad momentum spread.

Another possible future problem presents itself by considering an alternative way of cancelling the low-frequency self-field of the electron. This may be done by defining the wave packet state with the coherent photon contributions included by giving the electron some mass spread rather than momentum spread. In this case the electron may be thought of as not having a discrete energy-momentum spectrum for a single electron state because of the zero mass of photons. This problem is interesting since then the asymptotic conditions of field theory cannot be used to define the in- and out-states for particles of definite mass.

The mass spread idea may be applied in a different way to present another possible research problem. Instead of simply building the mass spread into the in- and out-states it may be possible to allow for uncertainty in mass by redefining the spectral representation for fermions. Using this redefined spectral representation and altering the LSZ reduction technique to allow a particle removed from the in- or out-state to have some mass uncertainty it should be possible to carry through the entire procedure to the point of obtaining Feynman diagrams and the corresponding rules.

Another possible research problem arises naturally because of its

analogous treatment to the photon infrared divergence problem. The resolution of the infrared divergence problem in graviton bremsstrahlung due to gravitational scattering should follow through in a similar manner. The graviton treatment can be expected to be more complicated because of the various possible gravitational interactions to be considered.

Fig. 1. Representation of basic diagrams for electron potential scattering containing all possible potential interactions, any real photons and n-1 virtual photons

Fig. 2. Representation of a second-order virtual photon radiative correction to electron potential scattering

Fig. 3. Representation of emission of a real photon from the incoming electron line

Fig. 4. Representation of emission of a real photon from the outgoing electron line

Fig. 5. Representation of ℓ noninteracting soft real photons, m soft real photons absorbed by the electron line and m' soft real photons emitted by the electron line

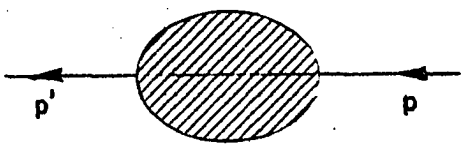

Fig. 1

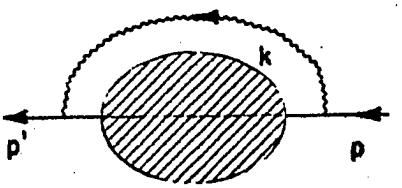

Fig. 2

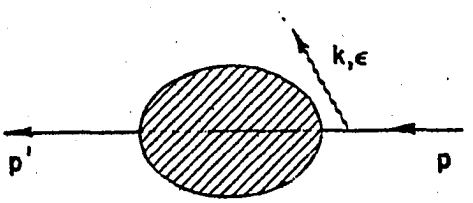

Fig. 3

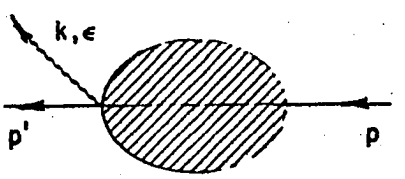

Fig. 4

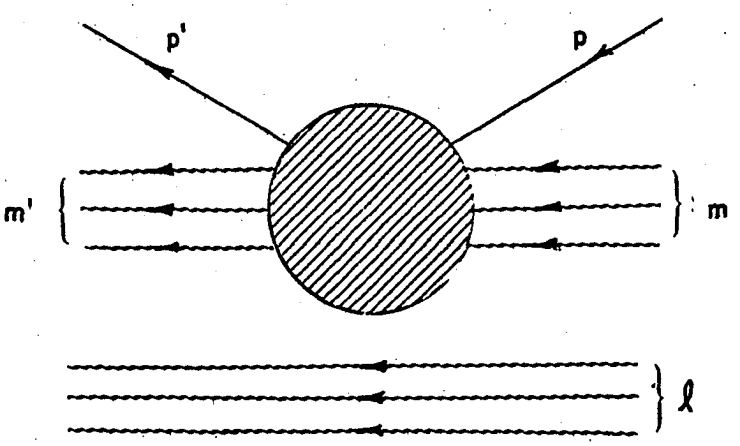

Fig. 5

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IX. APPENDIX

The space-time coordinates are denoted by the contravariant four-vector

$$X^{\mu} = (X^{o}, \vec{X}) = (ct, \vec{X}) = (ct, x, y, z) \tag{151}$$

The metric tensor $g_{\mu\nu}$ is defined with components

$$g_{oo} = -g_{11} = -g_{22} = -g_{33} = 1 \tag{152}$$

$$g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu \tag{153}$$

so the covariant four-vector is

$$X_{\mu} = (X^{o}, -\vec{X}) \tag{154}$$

The Greek indices run from 0 to 3 while the Latin indices represent the space components only. The summation convention is used so that

$$g_{\mu\nu}\, a^{\nu} = \sum_{\nu=0}^{3} g_{\mu\nu}\, a^{\nu} \tag{155}$$

The Lorentz invariant scalar product of two four-vectors is defined by

$$a \cdot b = a^{\mu} b_{\mu} = a_{o} b_{o} - \vec{a} \cdot \vec{b} \tag{156}$$

Feynman's slash notation is used and denotes the Lorentz invariant scalar product of an ordinary four-vector and a γ matrix,

$$\not{a} = a_{\mu}\gamma^{\mu} = a_{o}\gamma_{o} - \vec{a} \cdot \vec{\gamma} \tag{157}$$

The momentum operator in the coordinate representation is

$$p_{\mu} = i\frac{\partial}{\partial X^{\mu}} = i\partial_{\mu} \tag{158}$$

The natural system of units is used: $\hbar = c = 1$.